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Journal of Functional Analysis 255 (2008) 1724–1760

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Local operator spaces, unbounded operators and multinormed C^*-algebras

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Received 19 October 2007; accepted 4 July 2008

Available online 8 August 2008

Communicated by P. Delorme

Abstract

In this paper we propose a representation theorem for local operator spaces which extends Ruan's representation theorem for operator spaces. Based upon this result, we introduce local operator systems which are locally convex versions of the operator systems and prove Stinespring theorem for local operator systems. A local operator C^*-algebra is an example of a local operator system. Finally, we investigate the injectivity in both local operator space and local operator system senses, and prove locally convex version of the known result by Choi and Effros, that an injective local operator system possesses unique multinormed C^*-algebra structure with respect to the original involution and matrix topology.

Keywords: Local operator space; Matrix seminorm; Local operator system; Quantized domain

1. Introduction

The known [9, Theorem 2.3.5] representation theorem for operator spaces asserts that each abstract operator space V can be realized as a subspace of the space $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H. By realization we mean a matrix isometry $\Phi : V \to \mathcal{B}(H)$ of V onto the subspace $\Phi(V) \subseteq \mathcal{B}(H)$. This result plays a central role in the operator space theory. It allows one to have an abstract characterization of a linear space of bounded linear operators on a Hilbert space.

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2008.07.006

Physically well motivated, operator spaces can be thought as quantized normed spaces, where we have replaced functions by operators regarding classical normed spaces as abstract function spaces. Another motivation is observed by the dominating property in the noncommutative functional calculus problem [6], which confirms that (joint) spectral properties of elements of an operator algebra can be expressed in terms of matrices over the original algebra. The implementation of this proposal would lead to a reasonable joint spectral theory in an operator algebra.

To have more solid justification of the quantum physics and noncommutative function theory it is necessary to consider operator analogues of locally convex spaces too, that is, quantizations of polynormed spaces. That leads to linear spaces of unbounded Hilbert space operators or more generally, projective limits of operator spaces. In recent years, this theory has been developed by Effros, Webster and Winkler in [10,11,20] under the title "local operator spaces." The central and subtle result of their investigations is an operator version of the classical bipolar theorem [10, Proposition 4.1], which asserts that the double operator polar of an absolutely matrix convex and weakly closed matrix set coincides with itself (see below Theorem 6.1). This result allows one to have a fine description of a continuous matrix seminorm on a local operator space in terms of the matrix duality (see Section 6). It also provides a scale of possible quantizations of a polynormed space, namely the scale of min and max quantizations. The Krein–Milman theorem for local operator spaces was proposed by Webster and Winkler in [21].

In the present work we provide an intrinsic description of local operator spaces as above mentioned characterization for operator spaces, and investigate the injectivity in this framework proposing locally convex version of the known result by Choi and Effros [4]. We prove that each local operator space can be realized as a linear space of unbounded operators on a Hilbert space, that is, as a concrete local operator space. Namely, we fix a Hilbert space H and its upward filtered family of closed subspaces $\mathcal{E}=\{H_\alpha\}_{\alpha\in\Lambda}$ such that their union $\mathcal{D}$ is a dense subspace in H. We say that $\mathcal{E}$ is *a quantized domain in H with its union space $\mathcal{D}$*. The family $\mathcal{E}$ in turn determines an upward filtered family $\mathfrak{p}=\{P_\alpha\}_{\alpha\in\Lambda}$ of projections in $\mathcal{B}(H)$ onto the subspaces H_α, $\alpha\in\Lambda$. *The algebra of all noncommutative continuous functions on a quantized domain* $\mathcal{E}$ is defined as

$$C_{\mathcal{E}}(\mathcal{D})=\bigl\{T\in L(\mathcal{D})\colon\ TP_\alpha=P_\alpha TP_\alpha\in\mathcal{B}(H),\ \alpha\in\Lambda\bigr\},$$

where $L(\mathcal{D})$ is the associative algebra of all linear transformations on $\mathcal{D}$. Thus each $T\in C_{\mathcal{E}}(\mathcal{D})$ is an unbounded operator on H with domain $\mathcal{D}$ such that $T(H_\alpha)\subseteq H_\alpha$ and $T|H_\alpha\in\mathcal{B}(H_\alpha)$. Obviously, $C_{\mathcal{E}}(\mathcal{D})$ is a subalgebra in $L(\mathcal{D})$. Let us also introduce *the $*$-algebra*

$$C^*_{\mathcal{E}}(\mathcal{D})=\bigl\{T\in C_{\mathcal{E}}(\mathcal{D})\colon\ P_\alpha T\subseteq TP_\alpha,\ \alpha\in\Lambda\bigr\}$$

of all noncommutative continuous functions on $\mathcal{E}$. Each unbounded operator $T\in C^*_{\mathcal{E}}(\mathcal{D})$ has an unbounded dual $T^\bigstar$ such that $\mathcal{D}\subseteq\operatorname{dom}(T^\bigstar)$, $T^\bigstar(\mathcal{D})\subseteq\mathcal{D}$ and $T^*=T^\bigstar|\mathcal{D}\in C^*_{\mathcal{E}}(\mathcal{D})$. Moreover, the correspondence $T\mapsto T^*$ is an involution on $C^*_{\mathcal{E}}(\mathcal{D})$, thereby $C^*_{\mathcal{E}}(\mathcal{D})$ is a unital $*$-subalgebra (see Proposition 3.1) in $C_{\mathcal{E}}(\mathcal{D})$. The family of (matrix) seminorms $p^{(n)}_\alpha(v)=\|v|H^n_\alpha\|$, $v\in M_n(C_{\mathcal{E}}(\mathcal{D}))$, $n\in\mathbb{N}$, $\alpha\in\Lambda$, determines a (complete) local operator space structure on $C_{\mathcal{E}}(\mathcal{D})$. Moreover, $C_{\mathcal{E}}(\mathcal{D})$ equipped with the polynormed topology determined by the family $\{p^{(1)}_\alpha\colon\ \alpha\in\Lambda\}$ is a unital Arens–Michael (complete locally multiplicative) algebra whose $*$-subalgebra $C^*_{\mathcal{E}}(\mathcal{D})$ is a unital multinormed C^*-algebra.

A concrete local operator space on $\mathcal{D}$ is defined as a (local operator) subspace in $C_{\mathcal{E}}(\mathcal{D})$. The representation theorem proved in Section 7 asserts that each local operator space V is a concrete local operator space on a certain quantized domain up to a topological matrix isomorphism, that is, there is a topological matrix embedding $V \to C_{\mathcal{E}}(\mathcal{D})$. Moreover, if V is a Fréchet operator space then $V \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ for a certain quantized Fréchet domain $\mathcal{E}$ (that is, $\mathcal{E}$ is countable).

The result on operator realizations of a local operator space generalizes Ruan's representation theorem for operator spaces. To restore a natural connection between the local operator spaces and unital multinormed C^*-algebras as it was in the normed case we introduce in Section 4.2 *a local operator system* V as a unital self-adjoint subspace of the multinormed C^*-algebra $C^*_{\mathcal{E}}(\mathcal{D})$. Thus $I_{\mathcal{D}} \in V$ and $T^* \in V$ for each $T \in V$. Note that each local operator system is an inverse limit of (normed) operator systems. A $*$-subalgebra in $C^*_{\mathcal{E}}(\mathcal{D})$ called *a local operator algebra* presents another example of a local operator system.

An important role in local operator systems is played by *the local positivity*. We say that an element T of a local operator system V is *local hermitian* if $T = T^*$ on a certain subspace H_α, that is, $T|H_\alpha = T^*|H_\alpha = (T|H_\alpha)^*$ in $\mathcal{B}(H_\alpha)$. Respectively, an element $T \in V$ is said to be *local positive* if $T|H_\alpha \geqslant 0$ in $\mathcal{B}(H_\alpha)$ for some $\alpha \in \Lambda$. Further, let $V \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ and $W \subseteq C^*_{\mathcal{S}}(\mathcal{O})$ be local operator systems on quantized domains $\mathcal{E} = \{H_\alpha\}_{\alpha \in \Lambda}$ and $\mathcal{S} = \{K_\iota\}_{\iota \in \Omega}$ with their union spaces $\mathcal{D}$ and $\mathcal{O}$, respectively. A linear mapping $\varphi : V \to W$ is said to be *local matrix positive* if for each $\iota \in \Omega$ there corresponds $\alpha \in \Lambda$ such that $\varphi^{(n)}(v)|K^n_\iota \geqslant 0$ whenever $v|H^n_\alpha \geqslant 0$, and $\varphi^{(n)}(v)|K^n_\iota = 0$ if $v|H^n_\alpha = 0$, $v \in M_n(V)$, $n \in \mathbb{N}$, where $\varphi^{(n)} : M_n(V) \to M_n(W)$, $n \in \mathbb{N}$, are the canonical linear extensions of φ over all matrix spaces.

Based upon the local positivity concept we prove Stinespring theorem for local operator systems which involves the multinormed C^*-algebra $C^*_{\mathcal{E}}(\mathcal{D})$ over a quantized domain $\mathcal{E}$ instead of $\mathcal{B}(H)$. Namely, let A be a unital multinormed C^*-algebra with a defining family of C^*-seminorms $\{p_\alpha\colon \alpha \in \Lambda\}$, $\mathcal{E}$ a quantized domain in a Hilbert space H with its union space $\mathcal{D}$ and let $\varphi : A \to C^*_{\mathcal{E}}(\mathcal{D})$ be a unital local matrix positive mapping. Then there are a quantized domain $\mathcal{S}$ in a Hilbert space K containing $\mathcal{E}$ and a unital local contractive $*$-homomorphism $\pi : A \to C^*_{\mathcal{S}}(\mathcal{O})$ such that

$$\varphi(a) \subseteq P_H \pi(a) \quad \text{for all } a \in A,$$

where $\mathcal{O}$ is the union space of $\mathcal{S}$ and $P_H \in \mathcal{B}(K)$ is the projection onto H. This result allows us to have a description of a C^*-seminorm on a unital multinormed C^*-algebra in terms of contractive $*$-representations whose representation spaces have bounded Hilbert space dimensions. Using this fact, we prove a representation theorem for unital multinormed C^*-algebras. Thus each unital multinormed C^*-algebra A is a local operator C^*-algebra up to a topological $*$-isomorphism, that is, there is a topological (matrix) $*$-isomorphism $A \to C^*_{\mathcal{E}}(\mathcal{D})$ from A into the multinormed C^*-algebra $C^*_{\mathcal{E}}(\mathcal{D})$ of all noncommutative continuous functions on a certain quantized domain $\mathcal{E}$ with its union space $\mathcal{D}$. This is a GNS theorem for noncommutative multinormed C^*-algebras.

In Section 8, we investigate the injectivity in both local operator space and local operator system senses. We prove Arveson–Hahn–Banach–Webster theorem for local operator systems which involves the algebra $C^*_{\mathcal{E}}(\mathcal{D})$ over a quantized Fréchet domain $\mathcal{E}$, instead of $\mathcal{B}(H)$. Namely, let V be a local operator system and let W be its operator system subspace. Then each local matrix positive mapping $\varphi : W \to C^*_{\mathcal{E}}(\mathcal{D})$ has a local matrix positive extension

$$\begin{tikzcd} V \arrow[r, "\Phi"] & C^*_{\mathcal{E}}(\mathcal{D}) \\ \cup & \\ W \arrow[uur, "\varphi"] & \end{tikzcd}$$

that is, $C^*_{\mathcal{E}}(\mathcal{D})$ is an injective local operator system. It seems that the natural analog of $\mathcal{B}(H)$ in the local operator space theory should be the algebra $C^*_{\mathcal{E}}(\mathcal{D})$ over a quantized domain $\mathcal{E}$.

Finally, we prove that each injective local operator system is a unital multinormed C^*-algebra with respect to the original involution and matrix topology. In the normed case this result was proved in [4] by Choi and Effros. Here we are observing a serious gap between normed and locally convex versions. In the normed case the original matrix norm on an injective operator system V in $\mathcal{B}(H)$ is a C^*-norm with respect to the new multiplication created by a morphism-projection $\mathcal{B}(H) \to \mathcal{B}(H)$ onto V, which in turn implies that V is a C^*-algebra. But similar argument fails to be true for an injective local operator system $V \subseteq C^*_{\mathcal{E}}(\mathcal{D})$. The original matrix seminorms on $C^*_{\mathcal{E}}(\mathcal{D})$ are not C^*-seminorms on V with respect to the new multiplication determined by a morphism-projection (unital local matrix positive projection) $C^*_{\mathcal{E}}(\mathcal{D}) \to C^*_{\mathcal{E}}(\mathcal{D})$ onto V. To overcome this obstacle we introduce in Section 8.3 a new family of C^*-seminorms on V which is equivalent to the original one. As a result we prove that each injective local operator system possesses unique multinormed C^*-algebra structure with respect to the original involution and matrix topology.

Some of the obtained results have been announced in [7] and they are applied to the quantized moment problems in [8].

2. Preliminaries

The direct product of complex linear spaces E and F is denoted by $E \times F$ and we put E^k for the k-times product $E \times \cdots \times E$. The linear space of all linear transformations between E and F is denoted by $L(E, F)$, and we write $L(E)$ instead of $L(E, E)$. The identity operator on E is denoted by I_E. It is the unit of the associative algebra $L(E)$. Take $T \in L(E)$. The n-fold inflation $T^{\oplus n} = T \oplus \cdots \oplus T \in L(E^n)$ of T is acting as $(x_i)_i \mapsto (Tx_i)_i$, $x_i \in E$, $1 \leqslant i \leqslant n$. If T leaves invariant a subspace $F \subseteq E$, then $T|F$ denotes the restriction of T to F. The C^*-algebra of all bounded linear operators on a Hilbert space H is denoted by $\mathcal{B}(H)$. The domain of an unbounded operator T on H is denoted by $\operatorname{dom}(T)$. For an unbounded operators T and S on H we write $T \subseteq S$ if $\operatorname{dom}(T) \subseteq \operatorname{dom}(S)$ and $Tx = Sx$ for all $x \in \operatorname{dom}(T)$. If T is a densely defined operator on H then $T^\bigstar$ denotes its dual operator, thus $\langle Tx, y\rangle = \langle x, T^\bigstar y\rangle$ for all $x \in \operatorname{dom}(T)$, $y \in \operatorname{dom}(T^\bigstar)$, where $\langle \cdot,\cdot \rangle$ is the inner product in H.

The linear space of all $m \times n$-matrices $x = [x_{ij}]$ over a linear space E is denoted by $\mathbb{M}_{m,n}(E)$, and we set $\mathbb{M}_m(E) = \mathbb{M}_{m,m}(E)$, $\mathbb{M}_{m,n} = \mathbb{M}_{m,n}(\mathbb{C})$. Further, $\mathbb{M}(E)$ denotes the linear space of all infinite matrices $[x_{ij}]$ ($x_{ij} \in E$) where all but finitely many of x_{ij} are zeros. If $E = \mathbb{C}$ we write $\mathbb{M}$ instead of $\mathbb{M}(\mathbb{C})$. Each $\mathbb{M}_{m,n}(E)$ is a subspace in $\mathbb{M}(E)$ comprising those matrices $x = [x_{ij}]$ with $x_{ij} = 0$ whenever $i > m$ or $j > n$. Moreover, $\mathbb{M}(E) = \varinjlim \mathbb{M}_{m,n}(E)$ is the inductive limit of these subspaces. Note that $\mathbb{M}_{m,n}(L(E)) = L(E^n, E^m)$ up to a canonical identification. In particular, $\mathbb{M}_n(L(E)) = L(E^n)$. The space $\mathbb{M}_{m,n}(E)$ (respectively, $\mathbb{M}(E)$) equipped with a certain polynormed (or locally convex) topology, is denoted by $M_{m,n}(E)$ (respectively, $M(E)$). For instance, if $E = H$ is a Hilbert space then $\mathbb{M}_n(\mathcal{B}(H)) = \mathcal{B}(H^n) = M_n(\mathcal{B}(H))$. In particular, $M_{m,n}$ is the space $\mathbb{M}_{m,n}$ with the operator norm $\|\cdot\|$ between the canonical Hilbert spaces $\mathbb{C}^n$ and $\mathbb{C}^m$. Take $a \in \mathbb{M}_{m,s}$, $v \in \mathbb{M}_{s,t}(E)$ and $b \in \mathbb{M}_{t,n}$. The matrix product $avb \in \mathbb{M}_{m,n}(E)$ is defined by the usual way $avb = [\sum_{k,l} a_{ik} v_{kl} b_{lj}]$. The direct sum of matrices $v \in \mathbb{M}_{s,t}(E)$ and $w \in \mathbb{M}_{m,n}(E)$ is denoted by $v \oplus w \in \mathbb{M}_{s+m,t+n}(E)$. By a *matrix set* $\mathfrak{B}$ *in the matrix space* $\mathbb{M}(E)$ we mean a collection $\mathfrak{B} = (\mathfrak{b}_n)$ of subsets $\mathfrak{b}_n \subseteq \mathbb{M}_n(E)$, $n \in \mathbb{N}$. For matrix subsets $\mathfrak{B}$ and $\mathfrak{M}$ in $\mathbb{M}(E)$ we write $\mathfrak{B} \subseteq \mathfrak{M}$ whenever $\mathfrak{b}_n \subseteq \mathfrak{m}_n$ for all n. A matrix set $\mathfrak{B}$ in $\mathbb{M}(E)$ is said to be *absolutely matrix convex* if $\mathfrak{b}_m \oplus \mathfrak{b}_n \subseteq \mathfrak{b}_{m+n}$ and $a\mathfrak{b}_m b \subseteq \mathfrak{b}_n$ for all contractions $a \in M_{n,m}$, $b \in M_{m,n}$, $m, n \in \mathbb{N}$.

One can easily verify that each $\mathfrak{b}_n$ is an absolutely convex set in $\mathbb{M}_n(E)$ whenever $\mathfrak{B} = (\mathfrak{b}_n)$ is absolutely matrix convex.

A linear mapping $\varphi: E \to F$ has the canonical linear extensions $\varphi^{(n)}: \mathbb{M}_n(E) \to \mathbb{M}_n(F)$, $\varphi^{(n)}([x_{ij}]) = [\varphi(x_{ij})]$, $n \in \mathbb{N}$, over all matrix spaces. We also have a linear mapping $\varphi^{(\infty)}$: $\mathbb{M}(E) \to \mathbb{M}(F)$ such that $\varphi^{(\infty)}|\mathbb{M}_n(E) = \varphi^{(n)}$, $n \in \mathbb{N}$.

2.1. *Local operator spaces*

Now we introduce local operator spaces [9,10,20]. Let E be a linear space and let $p^{(n)}: \mathbb{M}_n(E) \to [0, \infty]$, $n \in \mathbb{N}$, be gauges (respectively, seminorms) over all matrix spaces. The family $p = (p^{(n)})_{n\in\mathbb{N}}$ is said to be a *matrix gauge* (*respectively, matrix seminorm*) [10] on E if p possesses the following properties:

M1. $p^{(m+n)}(v \oplus w) \leqslant \max\{p^{(m)}(v), p^{(n)}(w)\}$.
M2. $p^{(n)}(avb) \leqslant \|a\| p^{(m)}(v) \|b\|$,

for all $v = [v_{ij}] \in \mathbb{M}_m(E)$, $w = [w_{ij}] \in \mathbb{M}_n(E)$, $a \in M_{n,m}$, $b \in M_{m,n}$, $n, m \in \mathbb{N}$. Note that M2 implies that

$$p^{(1)}(v_{ij}) = p^{(1)}\big(E_i v E_j^*\big) \leqslant p^{(n)}(v) = p^{(n)}\Big(\sum E_i^* v_{ij} E_j\Big) \leqslant \sum p^{(1)}(v_{ij}) \tag{2.1}$$

for any $v = [v_{ij}] \in \mathbb{M}_m(E)$, where E_i are the canonical row matrices. Let p and q be matrix gauges on E. We write $p \preccurlyeq q$ whenever $p^{(n)} \leqslant q^{(n)}$ for all $n \in \mathbb{N}$. It is a partial order structure over all matrix gauges on E. In particular, we define $\sup_\iota p_\iota = \{\sup_\iota p_\iota^{(n)}\colon n \in \mathbb{N}\}$ for a family $\{p_\iota\}$ of matrix gauges on E. Note that for a matrix gauge p on E, M1 implies that we have a well-defined gauge $p^{(\infty)}: \mathbb{M}(E) \to [0, \infty]$ given by the rule $p^{(\infty)}(x) = p^{(n)}(x)$, $x \in \mathbb{M}_n(E)$, furthermore the relation $p \preccurlyeq q$ for matrix gauges turns out to be a usual relation $p^{(\infty)} \leqslant q^{(\infty)}$ between the gauges on $\mathbb{M}(E)$. If p is a matrix gauge then the corresponding $p^{(\infty)}$ is an $\mathbb{M}$-module gauge on $\mathbb{M}(E)$ [10], that is, $p^{(\infty)}(x + y) = \max\{p^{(\infty)}(x), p^{(\infty)}(y)\}$ for orthogonal elements $x, y \in \mathbb{M}(E)$, and $p^{(\infty)}(axb) \leqslant \|a\| p^{(\infty)}(x) \|b\|$ for all $a, b \in \mathbb{M}$, $x \in \mathbb{M}(E)$. Moreover, this correspondence is a one-to-one relation between the matrix gauges on E and $\mathbb{M}$-module gauges on $\mathbb{M}(E)$ [10]. If $\{p_\iota\}$ is a family of matrix seminorms on E then obviously $\sup_\iota p_\iota$ is a matrix gauge on E and $(\sup_\iota p_\iota)^\infty = \sup_\iota p_\iota^\infty$.

If $p = (p^{(n)})_{n\in\mathbb{N}}$ is a matrix gauge on a linear space E and $\mathfrak{B} = \{p \leqslant 1\}$ then $\mathfrak{B} = (\mathfrak{b}_n)$ is a matrix set in $\mathbb{M}(E)$ with $\mathfrak{b}_n = \{p^{(n)} \leqslant 1\}$, which is absolutely matrix convex. In particular, $\bigcup_n \mathfrak{b}_n$ is the unit set of the $\mathbb{M}$-module gauge $p^{(\infty)}$ on $\mathbb{M}(E)$. Conversely, if $\mathfrak{B} = (\mathfrak{b}_n)$ is an absolutely matrix convex set in $\mathbb{M}(E)$ and $\gamma^{(n)}$ is the Minkowski functional (on $\mathbb{M}_n(E)$) of $\mathfrak{b}_n$, then $\gamma = (\gamma^{(n)})_{n\in\mathbb{N}}$ is a matrix gauge on E [10]. We say that γ is *the Minkowski functional of* $\mathfrak{B}$.

A linear space E with a (separated) family of matrix seminorms $\{p_\alpha\colon \alpha \in \Lambda\}$ is called an *abstract local operator space*. Note that the local operator space structure on E determines a polynormed (Hausdorff) topology on $\mathbb{M}(E)$ by means of the family of seminorms $\{p_\alpha^{(\infty)}\colon \alpha \in \Lambda\}$. The relevant polynormed space is denoted by $M(E)$. A linear space E is said to be an (*abstract*) *operator space* if E is endowed with a matrix norm. Let E be a local operator space. Then each matrix space $\mathbb{M}_n(E)$ turns into a polynormed space (or normed space in the operator space case) denoted by $M_n(E)$ with a defining family of seminorms $\{p_\alpha^{(n)}\colon \alpha \in \Lambda\}$, that is, $M_n(E)$ is just a closed subspace in $M(E)$ (see (2.1)). The matrix seminorms $\{p_\alpha\colon \alpha \in \Lambda\}$ and

$\{q_\iota\colon \iota \in \Omega\}$ on the same space E are assumed to be equivalent if for each $\alpha \in \Lambda$ there correspond a finite subset $F \subseteq \Omega$ and a positive constant $C_{\alpha F}$ such that $p_\alpha \preccurlyeq C_{\alpha F} \sup\{q_\iota\colon \iota \in F\}$ and vice versa, that is, the family of seminorms $\{p_\alpha^{(\infty)}\colon \alpha \in \Lambda\}$ and $\{q_\iota^{(\infty)}\colon \iota \in \Omega\}$ on $M(E)$ are equivalent in a usual manner. By a defining matrix seminorm family we mean any matrix seminorm family that is equivalent to the original one. Obviously, all equivalent families of matrix seminorms define the same topology on $M(E)$, in particular, over all matrix spaces $M_n(E)$, which is just the direct-product topology inherited by means of the canonical identifications $\mathbb{M}_n(E) \cong E^{n^2}$ (see (2.1)), $n \in \mathbb{N}$. Given a defining family of matrix seminorms, one also has its saturation $\{\sup\{p_\alpha\colon \alpha \in F\}\colon F \subseteq \Lambda\}$, where F runs over all finite subsets. Note that the saturation is an upward filtered family of matrix seminorms which is equivalent to the original family. Thereby, when convenient, one can assume that the considered family of matrix seminorms is saturated.

Let E and F be local operator spaces with their (saturated) family of matrix seminorms $\{p_\alpha\colon \alpha \in \Lambda\}$ and $\{q_\iota\colon \iota \in \Omega\}$, respectively. A linear mapping $\varphi : E \to F$ is said to be a *matrix continuous* if for each $\iota \in \Omega$ there corresponds $\alpha \in \Lambda$ and a positive constant $C_{\iota\alpha}$ such that $q_\iota^{(\infty)}\varphi^{(\infty)} \leqslant C_{\iota\alpha} p_\alpha^{(\infty)}$. If φ is invertible and φ^{-1} is matrix continuous too then we say that φ is a *topological matrix isomorphism.* If $C_{\iota\alpha} \leqslant 1$ for all possible ι and α, then φ is called a *local matrix contraction* with respect to the families $\{p_\alpha\colon \alpha \in \Lambda\}$ and $\{q_\iota\colon \iota \in \Omega\}$. The matrix contractions between local operator systems will play an important role (see Section 5).

2.2. *Multinormed C^*-algebras*

Recall that a seminorm p on a unital associative algebra A is called a *multiplicative seminorm* if $p(1_A) = 1$ and $p(ab) \leqslant p(a)p(b)$ for all $a, b \in A$. A multiplicative seminorm on an associative $*$-algebra A is said to be a *C^*-seminorm* if $p(a^*) = p(a)$ and $p(a^*a) = p(a)^2$ for all $a \in A$. A complete polynormed algebra with a defining family of multiplicative seminorms (respectively, C^*-seminorms) is called an *Arens–Michael algebra* (respectively, *multinormed C^*-algebra*) [12, 1.2.4].

Let A be a unital multinormed C^*-algebra with a defining family $\{p_\alpha\colon \alpha \in \Lambda\}$ of C^*-seminorms. We introduce local positivity in A with respect to the family $\{p_\alpha\colon \alpha \in \Lambda\}$. An element $a \in A$ is *local hermitian* if $a = a^* + x$ for some $x \in A$ such that $p_\alpha(x) = 0$ for some $\alpha \in \Lambda$. The set of all local hermitian elements in A is denoted by A_{lh}. An element $a \in A$ is said to be *local positive* if $a = b^*b + x$ for some $x \in A$ such that $p_\alpha(x) = 0$ for some $\alpha \in \Lambda$. The set of all local positive elements in A is denoted by A^+. For each $\alpha \in \Lambda$, let A_α be the C^*-algebra associated with the C^*-seminorm p_α. If $p_\alpha \leqslant p_\beta$ then there is a canonical $*$-homomorphism $\pi_{\alpha\beta} : A_\beta \to A_\alpha$ such that $\pi_{\alpha\beta}\pi_\beta = \pi_\alpha$, where $\pi_\alpha : A \to A_\alpha$ is the canonical $*$-homomorphism associated with the quotient mapping. Thus A is the inverse limit of the projective system $\{A_\alpha, \pi_{\alpha\beta}\colon \alpha, \beta \in \Lambda\}$ of C^*-algebras [2,18]. One can easily verify that $a \in A_{\mathrm{lh}}$ (respectively, $a \in A^+$) iff $\pi_\alpha(a)$ is hermitian (respectively, positive) in A_α for some α. We write $a \geqslant_\alpha 0$ (respectively, $a =_\alpha 0$) if $\pi_\alpha(a)$ is positive in A_α (respectively, $\pi_\alpha(a) = 0$). Evidently, $1_A \in A^+ \subseteq A_{\mathrm{lh}}$. Moreover, $A_{\mathrm{lh}} \subseteq A^+ - A^+$. Indeed, if $a \in A_{\mathrm{lh}}$ then $\pi_\alpha(a)$ is hermitian for some α. But $a = (a + p_\alpha(a)1_A) - p_\alpha(a)1_A$, and $\pi_\alpha(a + p_\alpha(a)1_A) = \pi_\alpha(a) + p_\alpha(a)1_{A_\alpha} = \pi_\alpha(a) + \|\pi_\alpha(a)\|_{A_\alpha} 1_{A_\alpha} \geqslant 0$ in A_α, so, $a + p_\alpha(a)1_A \in A^+$.

It is worth to note that each multinormed C^*-algebra A has a canonical local operator space structure. Indeed, let $\{p_\alpha^{(1)}\colon \alpha \in \Lambda\}$ be a defining family of C^*-seminorms on A. Each C^*-algebra A_α associated with the C^*-seminorm p_α has the canonical operator space structure and let $\|\cdot\|_\alpha = (\|\cdot\|_\alpha^{(n)})_{n\in\mathbb{N}}$ be its matrix norm. Then $p_\alpha = \|\cdot\|_\alpha \pi_\alpha^{(\infty)}$(that is, $p_\alpha^{(n)} = \|\cdot\|_\alpha^{(n)} \pi_\alpha^{(n)}$,

$n \in \mathbb{N}$) is a matrix seminorm on A. Thus A is a local operator space with the defining family of matrix seminorms $\{p_\alpha\}$.

Let A and B be multinormed C^*-algebras with their canonical matrix seminorm families $\{p_\alpha\colon\ \alpha \in \Lambda\}$ and $\{q_\iota\colon\ \iota \in \Omega\}$ associated with the relevant C^*-seminorms. A linear mapping $\varphi : A \to B$ is said to be *local positive* with respect to the indicated families if $\varphi(a) \geqslant_\iota 0$ whenever $a \geqslant_\alpha 0$, and $\varphi(a) =_\iota 0$ if $a =_\alpha 0$, $a \in A$. For brevity, we write $\varphi(a) >_\iota 0$ whenever $a >_\alpha 0$. Further, a linear mapping $\varphi : A \to B$ is called *local matrix positive* if for each $\iota \in \Omega$ there corresponds $\alpha \in \Lambda$ such that $\varphi^{(n)}(a) >_\iota 0$ (that is, $\pi_\iota^{(n)}(\varphi^{(n)}(a)) > 0$ in $M_n(B_\iota)$) whenever $a >_\alpha 0$ (that is, $\pi_\alpha^{(n)}(a) > 0$ in $M_n(A_\alpha)$), $a \in M_n(A)$, $n \in \mathbb{N}$. In particular, all $\varphi^{(n)}$ are local positive mappings.

3. Unbounded operators over quantized domains

In this section we introduce quantized domains in a Hilbert space over which will be introduced noncommutative continuous functions or unbounded operators. That will lead to the concept of a concrete local operator space.

Fix a Hilbert space H and its dense subspace $\mathcal{D}$, and let V be a linear space of linear transformations on $\mathcal{D}$. Note that V can be thought as a linear space of unbounded operators on H whose elements have the same domain $\mathcal{D}$ and they leave invariant that domain. If $\mathcal{D}$ is closed, that is, $\mathcal{D} = H$, then $V \subseteq L(H)$. Recall that if $V \subseteq \mathcal{B}(H)$ is a linear subspace, then it can be equipped with the weak and strong operator topologies (WOT and SOT, respectively) determined by the seminorms $w_{x,y}(T) = |\langle Tx, y\rangle|$, $x, y \in H$, and $s_x(T) = \|Tx\|$, $x, y \in H$, respectively.

3.1. *The projection nets*

Let $\mathfrak{p} = \{P_\alpha\}_{\alpha\in\Lambda}$ be an upward filtered (or directed) set of (self-adjoint) projections in $\mathcal{B}(H)$, that is, $H_\alpha = \operatorname{im}(P_\alpha)$ is a closed subspace in H, P_α is the projection onto H_α along the orthogonal complemented subspace $H_\alpha^\perp$, and for a couple of projections P_α and P_β from $\mathfrak{p}$ there is another one P_γ ($\gamma \in \Lambda$) with $P_\alpha \leqslant P_\gamma$ and $P_\beta \leqslant P_\gamma$. That is, $P_\alpha = P_\alpha P_\gamma = P_\gamma P_\alpha$ and $P_\beta = P_\beta P_\gamma = P_\gamma P_\beta$, or $H_\alpha \subseteq H_\gamma$ and $H_\beta \subseteq H_\gamma$. We set $\alpha \leqslant \beta$ whenever $P_\alpha \leqslant P_\beta$, thereby Λ is a directed index set, and $\mathfrak{p}$ can be treated as a net of projections in $\mathcal{B}(H)$.

Now let $\mathfrak{p} = \{P_\alpha\}_{\alpha\in\Lambda}$ and $\mathfrak{q} = \{Q_\iota\}_{\iota\in\Omega}$ be projection nets in $\mathcal{B}(H)$. We write $\mathfrak{p} \preccurlyeq \mathfrak{q}$ if for each P_α one may find Q_ι such that $P_\alpha \leqslant Q_\iota$. The nets $\mathfrak{p}$ and $\mathfrak{q}$ are assumed to be *equivalent* ($\mathfrak{p} \sim \mathfrak{q}$) if $\mathfrak{p} \preccurlyeq \mathfrak{q}$ and $\mathfrak{q} \preccurlyeq \mathfrak{p}$.

The forthcoming assertion is well known [15, Theorem 4.1.2].

Lemma 3.1. *Let $\mathfrak{p} = \{P_\alpha\}_{\alpha\in\Lambda}$ be a projection net in $\mathcal{B}(H)$. There is a projection $P \in \mathcal{B}(H)$ such that $P_\alpha \to P$ (WOT). Moreover, $P_\alpha \to P$ (WOT) is equivalent to $P_\alpha \to P$ (SOT), and $P_\alpha \leqslant P$ for all α.*

We write $P = \lim \mathfrak{p}$ if $\mathfrak{p} = \{P_\alpha\}_{\alpha\in\Lambda}$ is a projection net and $P_\alpha \to P$ (WOT).

Corollary 3.1. *Let $\mathfrak{p} = \{P_\alpha\}_{\alpha\in\Lambda}$ be a projection net in $\mathcal{B}(H)$, $P = \lim \mathfrak{p}$ and let $H_\alpha = \operatorname{im}(P_\alpha)$, $\alpha \in \Lambda$. Then $\mathcal{D} = \bigcup_{\alpha\in\Lambda} H_\alpha$ is a subspace in H and $\overline{\mathcal{D}} = \operatorname{im}(P)$.*

Proof. Since $\{H_\alpha\}_{\alpha\in\Lambda}$ is an upward filtered family of subspaces in H, it follows that $\mathcal{D}$ is a linear subspace in H. By Lemma 3.1, $P_\alpha \leqslant P$, therefore $H_\alpha \subseteq \operatorname{im}(P)$ for all α. Consequently, $\overline{\mathcal{D}} \subseteq \operatorname{im}(P)$. Conversely, if $x \in \operatorname{im}(P)$ then $x = Px = \lim_\alpha P_\alpha x \in \overline{\mathcal{D}}$. Thus $\overline{\mathcal{D}} = \operatorname{im}(P)$. $\square$

Remark 3.1. If $\mathfrak{p} \preccurlyeq \mathfrak{q}$ then $\lim \mathfrak{p} \leqslant \lim \mathfrak{q}$. Moreover, the equivalent nets have the same limit.

3.2. Quantized domains

Let H be a Hilbert space. By *a quantized domain* (or merely *domain*) *in* H we mean an upward filtered family $\mathcal{E} = \{H_\alpha\}_{\alpha\in\Lambda}$ of closed subspaces in H whose union $\mathcal{D} = \bigcup \mathcal{E}$ is dense in H. Note that $\mathcal{D}$ is a linear subspace in H (see Corollary 3.1). We say that $\mathcal{D}$ is *the union space of the quantized domain* $\mathcal{D}$. If $\mathcal{E} = \{H_\alpha\}_{\alpha\in\Lambda}$ and $\mathcal{K} = \{K_\iota\}_{\iota\in\Omega}$ are domains in H then we write $\mathcal{E} \subseteq \mathcal{K}$ whenever $\Lambda = \Omega$ and $H_\alpha \subseteq K_\alpha$ for all $\alpha \in \Lambda$. Thus $\mathcal{E} = \mathcal{K}$ if and only if $\mathcal{E} \subseteq \mathcal{K}$ and $\mathcal{K} \subseteq \mathcal{E}$. Further, the domains $\mathcal{E} = \{H_\alpha\}_{\alpha\in\Lambda}$ and $\mathcal{K} = \{K_\iota\}_{\iota\in\Omega}$ in H are assumed to be equivalent $\mathcal{E} \sim \mathcal{K}$ if for each $\alpha \in \Lambda$ there corresponds $\iota \in \Omega$ with $H_\alpha \subseteq K_\iota$, and vice versa. Confirm that the equivalent domains have the same union space. In particular, the disjoint union $\mathcal{E} \vee \mathcal{K} = \{H_\alpha, K_\iota\}_{\alpha\in\Lambda,\iota\in\Omega}$ is a quantized domain in H with the same union space $\mathcal{D}$, whenever $\mathcal{E} \sim \mathcal{K}$. Each domain $\mathcal{E} = \{H_\alpha\}_{\alpha\in\Lambda}$ in H automatically associates a projection net $\mathfrak{p} = \{P_\alpha\}_{\alpha\in\Lambda}$ in $\mathcal{B}(H)$ over all subspaces H_α, $\alpha \in \Lambda$. Without any doubt $\mathcal{E} \sim \mathcal{K}$ iff $\mathfrak{p} \sim \mathfrak{q}$ for the domains $\mathcal{E}$ and $\mathcal{K}$ in H with their relevant projection nets $\mathfrak{p}$ and $\mathfrak{q}$. Moreover, $1_H = \lim \mathfrak{p}$ thanks to Lemma 3.1 and Corollary 3.1.

Let us introduce *the algebra of all noncommutative continuous functions on a quantized domain* $\mathcal{E} = \{H_\alpha\}_{\alpha\in\Lambda}$ *with its union space* $\mathcal{D}$ as

$$C_{\mathcal{E}}(\mathcal{D}) = \bigl\{T \in L(\mathcal{D}):\ TP_\alpha = P_\alpha T P_\alpha \in \mathcal{B}(H),\ \alpha \in \Lambda\bigr\}, \tag{3.1}$$

where $\mathfrak{p} = \{P_\alpha\}_{\alpha\in\Lambda}$ is the projection net associated with $\mathcal{E}$. Thus $T(H_\alpha) \subseteq H_\alpha$ and $T|H_\alpha \in \mathcal{B}(H_\alpha)$ whenever $T \in C_{\mathcal{E}}(\mathcal{D})$. Obviously, $C_{\mathcal{E}}(\mathcal{D})$ is a unital subalgebra in $L(\mathcal{D})$.

Remark 3.2. It is worth to note that the union space $\mathcal{D}$ can be regarded as a strict inductive limit of a directed Hilbert space family $\mathcal{E} = \{H_\alpha\}_{\alpha\in\Lambda}$, that is, $\mathcal{D} = \varinjlim \mathcal{E}$. It is a complete polynormed space and $T \in C_{\mathcal{E}}(\mathcal{D})$ leaves invariant each subspace $H_\alpha \subseteq \mathcal{D}$ and $T|H_\alpha \in \mathcal{B}(H_\alpha)$. Therefore T is continuous over the inductive limit [13, Lemma 5.2].

It is also important to introduce the following subalgebra

$$C^*_{\mathcal{E}}(\mathcal{D}) = \bigl\{T \in C_{\mathcal{E}}(\mathcal{D}):\ P_\alpha T \subseteq T P_\alpha,\ \alpha \in \Lambda\bigr\} \tag{3.2}$$

in $C_{\mathcal{E}}(\mathcal{D})$ called *the $*$-algebra of all noncommutative continuous functions on a quantized domain* $\mathcal{E}$. Actually, $C^*_{\mathcal{E}}(\mathcal{D})$ possesses the natural involution as follows from the following assertion.

Proposition 3.1. *Each unbounded operator $T \in C^*_{\mathcal{E}}(\mathcal{D})$ has an unbounded dual $T^\bigstar$ such that $\mathcal{D} \subseteq \operatorname{dom}(T^\bigstar)$, $T^\bigstar(\mathcal{D}) \subseteq \mathcal{D}$ and $T^* = T^\bigstar|\mathcal{D} \in C^*_{\mathcal{E}}(\mathcal{D})$. The correspondence $T \mapsto T^*$ is an involution on $C^*_{\mathcal{E}}(\mathcal{D})$, thereby $C^*_{\mathcal{E}}(\mathcal{D})$ is a unital $*$-algebra. Conversely, if an unbounded operator $T \in C_{\mathcal{E}}(\mathcal{D})$ admits an unbounded dual $T^\bigstar$ such that $T^* = T^\bigstar|\mathcal{D} \in C_{\mathcal{E}}(\mathcal{D})$ then $T \in C^*_{\mathcal{E}}(\mathcal{D})$. In particular, $C^*_{\mathcal{E}}(\mathcal{D})$ consists of closable unbounded operators.*

Proof. Taking into account that $P_\alpha T \subseteq T P_\alpha$ and $P_\alpha T P_\alpha \in \mathcal{B}(H)$ for all $\alpha \in \Lambda$, let us pick all dual operators $(T|H_\alpha)^\bigstar \in \mathcal{B}(H_\alpha)$, $\alpha \in \Lambda$. If $\alpha \leqslant \beta$ then $P_\alpha(T|H_\beta) = (T|H_\beta)(P_\alpha|H_\beta)$ which in turn implies that $P_\alpha(T|H_\beta)^\bigstar = (T|H_\beta)^\bigstar(P_\alpha|H_\beta)$, that is, $P_\alpha(T|H_\beta)^\bigstar \subseteq (T|H_\beta)^\bigstar P_\alpha$. Moreover,

$$\langle x, (T|H_\beta)^\bigstar y\rangle = \langle x, P_\alpha(T|H_\beta)^\bigstar y\rangle = \langle x, (T|H_\beta)^\bigstar P_\alpha y\rangle = \langle T|H_\beta x, y\rangle = \langle T|H_\alpha x, y\rangle$$
$$= \langle x, (T|H_\alpha)^\bigstar y\rangle,$$

$x, y \in H_\alpha$. Put $Sx = (T|H_\alpha)^\bigstar x$ if $x \in H_\alpha$. Evidently, $\operatorname{dom}(S) = \mathcal{D}$ and $S(\mathcal{D}) \subseteq \mathcal{D}$. Moreover, $\langle Tx, y\rangle = \langle T|H_\alpha x, y\rangle = \langle x, (T|H_\alpha)^\bigstar y\rangle = \langle x, Sy\rangle$ for all $x, y \in H_\alpha$. Whence T admits an unbounded dual $T^\bigstar$ such that $S = T^\bigstar|\mathcal{D}$, that is, $S = T^*$ and $P_\alpha S \subseteq S P_\alpha$ for all α. Thus $T^* \in C^*_{\mathcal{E}}(\mathcal{D})$ and the mapping $T \mapsto T^*$ is an involution on $C^*_{\mathcal{E}}(\mathcal{D})$.

Conversely, take $T \in C_{\mathcal{E}}(\mathcal{D})$ such that $T^\bigstar|\mathcal{D} \in C_{\mathcal{E}}(\mathcal{D})$. So, $\mathcal{D} \subseteq \operatorname{dom}(T^\bigstar)$, $T^\bigstar(H_\alpha) \subseteq H_\alpha$ for all $\alpha \in \Lambda$, and $\langle Tx, y\rangle = \langle x, T^\bigstar y\rangle$ for all $x, y \in \mathcal{D}$. Furthermore,

$$P_\alpha T \subseteq \left(T^\bigstar P_\alpha\right)^\bigstar = \left(P_\alpha T^\bigstar P_\alpha\right)^\bigstar = P_\alpha T P_\alpha = T P_\alpha.$$

Whence $T \in C^*_{\mathcal{E}}(\mathcal{D})$ (see (3.2)).

Finally, take $T \in C^*_{\mathcal{E}}(\mathcal{D})$, and assume that $\lim x_n = 0$ and $\lim T x_n = z$ for a certain sequence $\{x_n\}$ in $\mathcal{D}$. If $y \in \mathcal{D}$ then $\langle z, y\rangle = \lim\langle T x_n, y\rangle = \lim\langle x_n, T^* y\rangle = 0$, that is, $z \perp \mathcal{D}$. Being $\mathcal{D}$ a dense subspace, infer that $z = 0$. Whence T admits the closure. $\square$

The algebra $C^*_{\mathcal{E}}(\mathcal{D})$ can be treated as a quantized version of the (commutative) multinormed C^*-algebra $C(\mathbb{R}^n)$ of all complex continuous functions on $\mathbb{R}^n$ equipped with the compact-open topology (see [8]). Respectively, the space $\mathcal{D}$ can be referred as a quantized version of the locally compact space $\mathbb{R}^n$ exhausted by an increasing family of compact subsets.

Any Hilbert space H can be treated as a quantized domain $\mathcal{E} = \{H\}$. In this case $C_{\mathcal{E}}(\mathcal{D}) = C^*_{\mathcal{E}}(\mathcal{D}) = \mathcal{B}(H)$. Note also that the family of all finite-dimensional subspaces of a dense subspace in a Hilbert space is an example of a quantized domain. More nontrivial examples are given below.

Example 3.1. Let $H = L_2(\mathbb{R})$ be the Hilbert space of all square integrable complex-valued functions on the real line with respect to the Lebesgue measure. Consider the linear subspace $\mathcal{D} \subseteq L_2(\mathbb{R})$ of those functions with compact supports, that is, each such function is vanishing a.e. outside of a compact interval in $\mathbb{R}$. Evidently, $\mathcal{D}$ is a dense subspace in $L_2(\mathbb{R})$. Moreover, $\mathcal{D}$ is exhausted by a countable family $\mathcal{E} = \{H_n\}_{n\in\mathbb{N}}$ of closed subspaces, where

$$H_n = \left\{f \in L_2(\mathbb{R}):\ \operatorname{supp}(f) \subseteq [-n, n]\right\}.$$

Thus $\mathcal{E}$ is a quantized domain in H with its union space $\mathcal{D}$.

The known multiplication operator by the independent real variable t belongs to $C^*_{\mathcal{E}}(\mathcal{D})$ and it is not a closed operator, where $\mathcal{E}$ is the quantized domain considered in Example 3.1. Namely, consider the unbounded operator T on $L_2(\mathbb{R})$ such that $\operatorname{dom}(T) = \mathcal{D}$ and $(Tf)(t) = tf(t)$. Obviously, $T(H_n) \subseteq H_n$ and $\|T|H_n\| \leqslant n$ for all n. Thereby $T \in C_{\mathcal{E}}(\mathcal{D})$. Since $\langle Tf, g\rangle = \langle f, Tg\rangle$, $f, g \in H_n$, it follows that T is a symmetric operator and $T^* = T^\bigstar|\mathcal{D} = T \in C_{\mathcal{E}}(\mathcal{D})$. Using

Proposition 3.1, we conclude that $T \in C^*_{\mathcal{E}}(\mathcal{D})$. Finally, T is not closed, just consider the family $f_n(t) = t^{-2}\chi_{(1,n)}(t)$, $n \in \mathbb{N}$, in $\mathcal{D}$, where $\chi_{(1,n)}$ is the indicator function of the interval $(1, n)$ (see [16, 13.13.16]).

Proposition 3.2. *Let $\mathcal{E}$ be a quantized domain in a Hilbert space H with its union space $\mathcal{D}$. If $\{\mathcal{K}_\theta\}$ is a family of quantized domains in H such that $\mathcal{K}_\theta \sim \mathcal{E}$ for all θ, then their disjoint union $\mathcal{K} = \bigvee\{\mathcal{K}_\theta\}$ is a domain in H such that $\mathcal{K} \sim \mathcal{E}$ and $\bigcap_\theta C_{\mathcal{K}_\theta}(\mathcal{D}) = C_{\mathcal{K}}(\mathcal{D})$, $\bigcap_\theta C^*_{\mathcal{K}_\theta}(\mathcal{D}) = C^*_{\mathcal{K}}(\mathcal{D})$. Moreover, $\bigcap_{\mathcal{E}} C_{\mathcal{E}}(\mathcal{D}) = \bigcap_{\mathcal{E}} C^*_{\mathcal{E}}(\mathcal{D}) = C_{\mathcal{F}}(\mathcal{D}) = C^*_{\mathcal{F}}(\mathcal{D}) = \mathbb{C}1_{\mathcal{D}}$, where $\mathcal{F}$ is the family of all finite-dimensional subspaces of $\mathcal{D}$.*

Proof. First note that if $T \in C_{\mathcal{F}}(\mathcal{D})$ then $Tx = \lambda_x x$ for each nonzero $x \in \mathcal{D}$. Being T a linear operator, we conclude that $T \in \mathbb{C}1_{\mathcal{D}}$. In particular, $C_{\mathcal{F}}(\mathcal{D}) = C^*_{\mathcal{F}}(\mathcal{D}) = \mathbb{C}1_{\mathcal{D}}$. It follows that $\bigcap_{\mathcal{E}} C_{\mathcal{E}}(\mathcal{D}) = C_{\mathcal{F}}(\mathcal{D}) = C^*_{\mathcal{F}}(\mathcal{D}) = \bigcap_{\mathcal{E}} C^*_{\mathcal{E}}(\mathcal{D}) = \mathbb{C}1_{\mathcal{D}}$. The rest follows from (3.1) and (3.2). □

Now fix a positive integer n and consider the nth Hilbert space power H^n of the Hilbert space H. If $\mathcal{E} = \{H_\alpha\}_{\alpha\in\Lambda}$ is a quantized domain in H with its union space $\mathcal{D}$ then so is $\mathcal{E}^n = \{H^n_\alpha\}_{\alpha\in\Lambda}$ in H^n whose union space is $\mathcal{D}^n$. If $\mathfrak{p} = \{P_\alpha\}_{\alpha\in\Lambda}$ is a projection net associated with $\mathcal{E}$ then so is $\mathfrak{p}^{\oplus n} = \{P^{\oplus n}_\alpha\}$ associated with $\mathcal{E}^n$. Indeed, $\mathfrak{p}^{\oplus n}$ consists of projections, namely $P^{\oplus n}_\alpha$ is the projection onto H^n_α ($H^{n\perp}_\alpha = H^{\perp n}_\alpha$), $\alpha \in \Lambda$. Fix $\alpha, \beta \in \Lambda$, $\alpha \leqslant \beta$. Then

$$\left\| P^{\oplus n}_\alpha x \right\|^2 = \left\| (P_\alpha x_i)_i \right\|^2 = \sum_{i=1}^n \| P_\alpha x_i \|^2 \leqslant \sum_{i=1}^n \| P_\beta x_i \|^2 = \left\| P^{\oplus n}_\beta x \right\|^2,$$

where $x = (x_i)_i \in H^n$. Thus $P^{\oplus n}_\alpha \leqslant P^{\oplus n}_\beta$. Hence $\mathfrak{p}^{\oplus n}$ is a projection net in $\mathcal{B}(H^n)$. By Lemma 3.1, there is $\lim \mathfrak{p}^{\oplus n}$ which we denote by Q. Taking into account that $P^{\oplus n}_\alpha \to Q$ (SOT) and $\lim_\alpha \|x - P^{\oplus n}_\alpha x\|^2 = \lim_\alpha \sum_{i=1}^n \|x_i - P_\alpha x_i\|^2 = \sum_{i=1}^n \lim_\alpha \|x_i - P_\alpha x_i\|^2 = 0$, we deduce that $Q = 1_{H^n}$.

Proposition 3.3. *For each $n \in \mathbb{N}$ we have*

$$\mathbb{M}_n\bigl(C_{\mathcal{E}}(\mathcal{D})\bigr) = C_{\mathcal{E}^n}\bigl(\mathcal{D}^n\bigr) \quad \textit{and} \quad \mathbb{M}_n\bigl(C^*_{\mathcal{E}}(\mathcal{D})\bigr) = C^*_{\mathcal{E}^n}\bigl(\mathcal{D}^n\bigr).$$

Proof. Take $T = [T_{ij}] \in \mathbb{M}_n(C_{\mathcal{E}}(\mathcal{D}))$. Then $T \in L(\mathcal{D}^n)$ and it leaves invariant each subspace H^n_α. Indeed, if $x = (x_i)_i \in H^n_\alpha$ then $Tx = (\sum_j T_{ij}x_j)_i \in H^n_\alpha$, for all $T_{ij}(H_\alpha) \subseteq H_\alpha$. Moreover, $T|H^n_\alpha = [T_{ij}|H_\alpha] \in M_n(\mathcal{B}(H_\alpha)) = \mathcal{B}(H^n_\alpha)$. Consequently, $T \in C_{\mathcal{E}^n}(\mathcal{D}^n)$. If $T \in \mathbb{M}_n(C^*_{\mathcal{E}}(\mathcal{D}))$ then $P^{\oplus n}_\alpha T = [P_\alpha T_{ij}] \subseteq [T_{ij} P_\alpha] = T P^{\oplus n}_\alpha$. Moreover, $P^{\oplus n}_\alpha T P^{\oplus n}_\alpha = [P_\alpha T_{ij} P_\alpha] \in \mathcal{B}(H^n)$. Whence $T \in C^*_{\mathcal{E}^n}(\mathcal{D}^n)$.

Conversely, take $T = [T_{ij}] \in C_{\mathcal{E}^n}(\mathcal{D}^n)$ (respectively, $T \in C^*_{\mathcal{E}^n}(\mathcal{D}^n)$). Since $T(H^n_\alpha) \subseteq H^n_\alpha$ (respectively, $[P_\alpha T_{ij}] \subseteq [T_{ij} P_\alpha]$), it follows that $T_{ij}(H_\alpha) \subseteq H_\alpha$ (respectively, $P_\alpha T_{ij} \subseteq T_{ij} P_\alpha$) for all i, j. Furthermore, $P_\alpha T_{ij} P_\alpha \in \mathcal{B}(H)$, for $P^{\oplus n}_\alpha T P^{\oplus n}_\alpha = [P_\alpha T_{ij} P_\alpha] \in \mathcal{B}(H^n)$. □

Remark 3.3. Note that if V is a $*$-subspace in $C^*_{\mathcal{E}}(\mathcal{D})$ (see Proposition 3.1) then $\mathbb{M}_n(V)$ is a $*$-subspace in $C^*_{\mathcal{E}^n}(\mathcal{D}^n)$. Indeed, $\mathbb{M}_n(V)$ is a linear subspace in $C^*_{\mathcal{E}^n}(\mathcal{D}^n)$ thanks to Proposition 3.3. Moreover, if $T = [T_{ij}] \in \mathbb{M}_n(V)$ then $T^* = [T^*_{ji}] \in C^*_{\mathcal{E}^n}(\mathcal{D}^n)$ by virtue of Proposition 3.1. But $T^*_{ji} \in V$ for all i, j. Whence $T^* \in \mathbb{M}_n(V)$.

4. Concrete local operator spaces and local operator systems

In this section we introduce a concrete local operator space as a subspace of the Arens–Michael algebra $C_{\mathcal{E}}(\mathcal{D})$ of all noncommutative continuous functions over a quantized domain $\mathcal{E}$, whereas the local operator systems (respectively, algebras) are unital self-adjoint subspaces (respectively, subalgebras) in the multinormed C^*-algebra $C^*_{\mathcal{E}}(\mathcal{D})$.

4.1. The matrix topology in $C_{\mathcal{E}}(\mathcal{D})$

Recall that a *concrete operator space E* is defined as a subspace of $\mathcal{B}(H)$ for a certain Hilbert space H (see [9, 2.1]). The inclusions $\mathbb{M}_n(E) \subseteq M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)$, $n \in \mathbb{N}$, determine the matrix norm on E.

Now let $\mathcal{E} = \{H_\alpha\}_{\alpha\in\Lambda}$ be a quantized domain in a Hilbert space H with its union space $\mathcal{D}$, and let $\mathfrak{p} = \{P_\alpha\}_{\alpha\in\Lambda}$ be the projection net in $\mathcal{B}(H)$ associated with $\mathcal{E}$ (see Section 3.2). Fix a positive integer n and take $T \in \mathbb{M}_n(C_{\mathcal{E}}(\mathcal{D}))$. By Proposition 3.3, $\mathbb{M}_n(C_{\mathcal{E}}(\mathcal{D})) = C_{\mathcal{E}^n}(\mathcal{D}^n)$. Thus T leaves invariant each subspace H^n_α and $\|T|H^n_\alpha\|_{\mathcal{B}(H^n_\alpha)} = \|P^{\oplus n}_\alpha T P^{\oplus n}_\alpha\|$. Put $p_\alpha = (p^{(n)}_\alpha)_{n\in\mathbb{N}}$, where $p^{(n)}_\alpha(T) = \|P^{\oplus n}_\alpha T P^{\oplus n}_\alpha\|$, $T \in \mathbb{M}_n(C_{\mathcal{E}}(\mathcal{D}))$, $\alpha \in \Lambda$.

Lemma 4.1. *The family $\{p_\alpha\colon \alpha \in \Lambda\}$ is an upward filtered family of matrix seminorms on $C_{\mathcal{E}}(\mathcal{D})$, which defines a matrix topology on $C_{\mathcal{E}}(\mathcal{D})$. If $\mathcal{E} \sim \mathcal{K}$ for some domain $\mathcal{K}$ in H then both matrix topologies on $C_{\mathcal{E}}(\mathcal{D})$ and $C_{\mathcal{K}}(\mathcal{D})$ coincide on $C_{\mathcal{E}\cup\mathcal{K}}(\mathcal{D})$. Moreover, $\{p^{(1)}_\alpha\colon \alpha \in \Lambda\}$ are multiplicative seminorms on $C_{\mathcal{E}}(\mathcal{D})$, which are C^*-seminorms on the $*$-subalgebra $C^*_{\mathcal{E}}(\mathcal{D})$.*

Proof. One can easily verify that each p_α is a matrix seminorm on $C_{\mathcal{E}}(\mathcal{D})$ and $p_\alpha \preccurlyeq p_\beta$ whenever $\alpha \leqslant \beta$. Consequently, $\{p_\alpha\colon \alpha \in \Lambda\}$ is an upward filtered family of matrix seminorms on $C_{\mathcal{E}}(\mathcal{D})$ and therefore it determines a matrix topology on $C_{\mathcal{E}}(\mathcal{D})$.

Now assume that $\mathcal{E} \sim \mathcal{K}$ for some domain $\mathcal{K} = \{K_\iota\}_{\iota\in\Omega}$ in H, and let $\mathfrak{p} = \{P_\alpha\}$ and $\mathfrak{q} = \{Q_\iota\}$ be the projection nets in $\mathcal{B}(H)$ associated with $\mathcal{E}$ and $\mathcal{K}$, respectively, which in turn involves the matrix seminorms $\mathfrak{P} = \{p_\alpha\}$ and $\mathfrak{Q} = \{q_\iota\}$, respectively. As we have confirmed in Section 3.2, $\mathcal{E} \sim \mathcal{K}$ iff $\mathfrak{p} \sim \mathfrak{q}$. By Proposition‘3.2, $C_{\mathcal{E}}(\mathcal{D}) \cap C_{\mathcal{K}}(\mathcal{D}) = C_{\mathcal{E}\vee\mathcal{K}}(\mathcal{D})$. It remains to prove that $\mathfrak{P}$ and $\mathfrak{Q}$ are equivalent matrix seminorm families ($\mathfrak{P} \sim \mathfrak{Q}$) on $C_{\mathcal{E}\vee\mathcal{K}}(\mathcal{D})$. By assumption, for each P_α there corresponds Q_ι such that $P_\alpha \leqslant Q_\iota$ and vice versa. It follows that $P_\alpha = P_\alpha Q_\iota = Q_\iota P_\alpha$ and using Proposition 3.3, we deduce that

$$\begin{aligned} p^{(n)}_\alpha(T) &= \left\| P^{\oplus n}_\alpha T P^{\oplus n}_\alpha \right\| = \left\| (P_\alpha Q_\iota)^{\oplus n} T P^{\oplus n}_\alpha \right\| = \left\| P^{\oplus n}_\alpha Q^{\oplus n}_\iota T P^{\oplus n}_\alpha \right\| = \left\| P^{\oplus n}_\alpha Q^{\oplus n}_\iota T Q^{\oplus n}_\iota P^{\oplus n}_\alpha \right\| \\ &\leqslant \left\| Q^{\oplus n}_\iota T Q^{\oplus n}_\iota \right\| = q^{(n)}_\iota(T) \end{aligned}$$

for all $T \in C_{\mathcal{E}^n}(\mathcal{D}^n) \cap C_{\mathcal{K}^n}(\mathcal{D}^n)$, $n \in \mathbb{N}$, that is, $p_\alpha \preccurlyeq q_\iota$. The rest is clear. $\square$

Thus $C_{\mathcal{E}}(\mathcal{D})$ has a canonical local operator space structure given by the family $\{p_\alpha\colon \alpha \in \Lambda\}$ of matrix seminorms associated with the domain $\mathcal{E}$. The next assertion states that $C_{\mathcal{E}}(\mathcal{D})$ is complete and its $*$-subalgebra $C^*_{\mathcal{E}}(\mathcal{D})$ is closed, that is, $C^*_{\mathcal{E}}(\mathcal{D})$ is a multinormed C^*-algebra.

Proposition 4.1. *The local operator space $C_{\mathcal{E}}(\mathcal{D})$ is complete. Thus $C_{\mathcal{E}}(\mathcal{D})$ is an Arens–Michael algebra whose $*$-subalgebra $C^*_{\mathcal{E}}(\mathcal{D})$ is a multinormed C^*-algebra.*

Proof. Consider a linear subspace $V \subseteq C_{\mathcal{E}}(\mathcal{D})$ and let $\widetilde{V}$ be its completion. We prove that $\widetilde{V}$ is identified up to a topological isomorphism with a subspace in $C_{\mathcal{E}}(\mathcal{D})$. Take a Cauchy net $\{T_\lambda\}$ in V. According to the definition, $\{T_\lambda|H_\alpha\}$ is a Cauchy net in $\mathcal{B}(H_\alpha)$ for each $\alpha \in \Lambda$. Put $T^{(\alpha)}$ for the (uniform) limit of the net $\{T_\lambda|H_\alpha\}$ in $\mathcal{B}(H_\alpha)$. If $\alpha \leqslant \beta$ then

$$T^{(\beta)}|H_\alpha = \Bigl(\lim_\lambda\{T_\lambda|H_\beta\}\Bigr)|H_\alpha = \lim_\lambda\bigl\{(T_\lambda|H_\beta)|H_\alpha\bigr\} = \lim_\lambda\{T_\lambda|H_\alpha\} = T^{(\alpha)}.$$

Thus we have a well-defined unbounded operator T on H such that $\operatorname{dom}(T) = \mathcal{D}$ and $T|H_\alpha = T^{(\alpha)}$ for all $\alpha \in \Lambda$. In particular, $T \in C_{\mathcal{E}}(\mathcal{D})$. One can easily verify that the linear mapping $\widetilde{V} \to C_{\mathcal{E}}(\mathcal{D})$, $\{T_\lambda\} \to T$, determines an embedding. Therefore $\widetilde{V}$ can be identified with a linear subspace in $C_{\mathcal{E}}(\mathcal{D})$. The family of seminorms $q_\alpha^{(n)}(T) = \|T|H_\alpha^n\|$, $T \in \mathbb{M}_n(\widetilde{V})$, determines a Hausdorff matrix polynormed topology on $\widetilde{V}$. Moreover, each $q_\alpha^{(1)}$ is a continuous seminorm on $\widetilde{V}$, for $q_\alpha^{(1)} = p_\alpha^{(1)}$ on the dense subspace V. So, the original uniformity on $\widetilde{V}$ dominates the uniformity associated with the family of seminorms $\{q_\alpha^{(1)}\}_{\alpha\in\Lambda}$. But both uniformities determine the same topology on the dense subspace V. It follows that these uniformities coincide [1, 2.3.14]. Whence $\widetilde{V}$ is identified with a subspace in $C_{\mathcal{E}}(\mathcal{D})$ up to a topological isomorphism. In particular, putting $V = C_{\mathcal{E}}(\mathcal{D})$ we derive that $C_{\mathcal{E}}(\mathcal{D})$ is complete. Bearing in mind (Lemma 4.1) that the defining seminorms $\{p_\alpha^{(1)}\}$ are multiplicative, we deduce that $C_{\mathcal{E}}(\mathcal{D})$ is an Arens–Michael algebra.

Now assume that $T \in C_{\mathcal{E}}(\mathcal{D})$ belongs to the closure of the $*$-subalgebra $C_{\mathcal{E}}^*(\mathcal{D})$ in $C_{\mathcal{E}}(\mathcal{D})$. So, there is a net $\{T_\lambda\}$ in $C_{\mathcal{E}}^*(\mathcal{D})$ such that $\lim\{T_\lambda\} = T$ in $C_{\mathcal{E}}(\mathcal{D})$. The latter means that $\lim\{T_\lambda|H_\alpha\} = T|H_\alpha$ for all α. Using Proposition 3.1, infer that $(T|H_\alpha)^\bigstar = (\lim\{T_\lambda|H_\alpha\})^\bigstar = \lim\{(T_\lambda|H_\alpha)^\bigstar\} = \lim\{T_\lambda^*|H_\alpha\}$, where $(T|H_\alpha)^\bigstar$ is the norm dual of $T|H_\alpha \in \mathcal{B}(H_\alpha)$. Put $Sx = (T|H_\alpha)^\bigstar x$ if $x \in H_\alpha$. It is a well-defined unbounded operator on H. Indeed, if $\alpha, \beta \in \Lambda$ with $\alpha \leqslant \beta$, then $(T|H_\beta)^\bigstar|H_\alpha = (\lim\{T_\lambda^*|H_\beta\})|H_\alpha = \lim\{(T_\lambda^*|H_\beta)|H_\alpha\} = \lim\{T_\lambda^*|H_\alpha\} = (T|H_\alpha)^\bigstar$. Therefore $Sx = (T|H_\alpha)^\bigstar x = (T|H_\beta)^\bigstar x$ if $x \in H_\alpha$. Moreover, $\operatorname{dom}(S) = \mathcal{D}$. Further, note that $\langle Tx, y\rangle = \langle x, (T|H_\alpha)^\bigstar y\rangle = \langle x, Sy\rangle$ for all $x, y \in H_\alpha$. It follows that $\langle Tx, y\rangle = \langle x, Sy\rangle$ for all $x, y \in \mathcal{D}$, that is, $S = T^\bigstar|\mathcal{D} = T^* \in C_{\mathcal{E}}(\mathcal{D})$. By Proposition 3.1, $T \in C_{\mathcal{E}}^*(\mathcal{D})$. Hence $C_{\mathcal{E}}^*(\mathcal{D})$ is a closed subalgebra in $C_{\mathcal{E}}(\mathcal{D})$. But all defining seminorms $\{p_\alpha^{(1)}\}$ on $C_{\mathcal{E}}(\mathcal{D})$ restricted to the $*$-algebra $C_{\mathcal{E}}^*(\mathcal{D})$ are C^*-seminorms (see Lemma 4.1), therefore $C_{\mathcal{E}}^*(\mathcal{D})$ is a unital multinormed C^*-algebra. $\square$

Now consider the Arens–Michael algebra $C_{\mathcal{E}}(\mathcal{D})$ over a quantized Fréchet domain $\mathcal{E} = \{H_n\}_{n\in\mathbb{N}}$, so $C_{\mathcal{E}}(\mathcal{D})$ is the Fréchet-Arens–Michael algebra. Let $\mathfrak{p} = \{P_n\}_{n\in\mathbb{N}}$ be the projection net associated with $\mathcal{E}$, and let $S_n = (1 - P_{n-1})P_n$ be the projection onto the subspace $H_{n-1}^\perp \cap H_n$, $n \geqslant 2$. For $n = 1$ we put $S_1 = P_1$.

The following assertion will be used later in Section 8.

Proposition 4.2. *If $T \in C_{\mathcal{E}}(\mathcal{D})$ then it has a triangular matrix representation*

$$T = \sum_{m=1}^{\infty}\sum_{k=1}^{m} S_k T S_m = \begin{bmatrix} T_{11} & T_{12} & \cdots \\ 0 & T_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

where $T_{km} = S_k T S_m \in \mathcal{B}(H)$, $k \leqslant m$. Moreover, if $T \in C^*_{\mathcal{E}}(\mathcal{D})$ then it has a diagonal representation $T = \sum_{m=1}^{\infty} S_m T S_m$, and the correspondence $T \mapsto \sum_{m=1}^{\infty} S_m T S_m$ implements a local matrix contractive projection $D : C_{\mathcal{E}}(\mathcal{D}) \to C_{\mathcal{E}}(\mathcal{D})$ onto the multinormed C^*-algebra $C^*_{\mathcal{E}}(\mathcal{D})$.

Proof. Take $T \in C_{\mathcal{E}}(\mathcal{D})$. Then $T P_n = P_n T P_n \in \mathcal{B}(H)$ for all n (see (3.1)). It follows that

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{k=1}^{m} S_k T S_m P_n &= \sum_{m=1}^{n}\sum_{k=1}^{m} S_k T S_m P_n = \sum_{m=1}^{n}\sum_{k=1}^{m} S_k T S_m = \sum_{m=1}^{n}\left(\sum_{k=1}^{m} S_k\right) T S_m \\
&= \sum_{m=1}^{n}\left(P_1 + \sum_{k=2}^{m}(1 - P_{k-1})P_k\right) T S_m = \sum_{m=1}^{n} P_m T S_m = \sum_{m=1}^{n} P_m T P_m S_m \\
&= \sum_{m=1}^{n} T P_m S_m = \sum_{m=1}^{n} T S_m \\
&= T P_n
\end{aligned}$$

for all n, that is, $T = \sum_{m=1}^{\infty}\sum_{k=1}^{m} S_k T S_m$. Thus T is a triangular operator given by the matrix $[T_{km}]_{k \leqslant m}$. If $T \in C^*_{\mathcal{E}}(\mathcal{D})$ then $S_k T S_m = (1 - P_{k-1})P_k T S_m = (1 - P_{k-1}) T P_k S_m = 0$ if $k < m$. Hence T is a diagonal operator and $T = \sum_{m=1}^{\infty} S_m T S_m = [T_{mm}]$.

Further, note that $P_n D(T) = P_n \sum_{k=1}^{\infty} S_k T S_k \subseteq P_n \sum_{k=1}^{n} S_k T S_k = \sum_{k=1}^{n} S_k T S_k = D(T) P_n$, for $P_i \leqslant P_{i+1}$ for all i. Thus $D(T)(H_n) \subseteq H_n$ and $D(T)|H_n = \bigoplus_{k=1}^{n} S_k T S_k \in \mathcal{B}(H_n)$ for all n, that is, D is well defined. Moreover, $D(T) \in C^*_{\mathcal{E}}(\mathcal{D})$.

Let us verify that $D : C_{\mathcal{E}}(\mathcal{D}) \to C_{\mathcal{E}}(\mathcal{D})$ is a local matrix contraction. First note that $D^{(s)}(v) = \sum_{n=1}^{\infty} S_n^{\oplus s} v S_n^{\oplus s}$ for all $v \in C_{\mathcal{E}^s}(\mathcal{D}^s)$. Then

$$\begin{aligned}
p_n^{(s)}\big(D^{(s)}(v)\big) &= \big\| P_n^{\oplus s} D^{(s)}(v) P_n^{\oplus s} \big\| = \left\| \bigoplus_{k=1}^{n} S_k^{\oplus s} v S_k^{\oplus s} \right\| = \max\big\{ \big\| S_k^{\oplus s} v S_k^{\oplus s} \big\| \colon k \leqslant n \big\} \\
&= \max\big\{ \big\| S_k^{\oplus s} P_n^{\oplus s} v P_n^{\oplus s} S_k^{\oplus s} \big\| \colon k \leqslant n \big\} \leqslant \big\| P_n^{\oplus s} v P_n^{\oplus s} \big\| = p_n^{(s)}(v).
\end{aligned}$$

Thus $p_n^{(\infty)} D^{(\infty)} \leqslant p_n^{(\infty)}$ for all n, thereby D is a local matrix contraction.

It remains to confirm that $D(T) = T$ whenever $T \in C^*_{\mathcal{E}}(\mathcal{D})$. Consequently, $D : C_{\mathcal{E}}(\mathcal{D}) \to C_{\mathcal{E}}(\mathcal{D})$ is a local matrix contractive projection onto $C^*_{\mathcal{E}}(\mathcal{D})$. □

4.2. Concrete models

Now we introduce concrete local operator spaces, local operator systems and local operator algebras as relevant subspaces in $C_{\mathcal{E}}(\mathcal{D})$ compatible with its interior structures.

If V is a linear subspace in $C^*_{\mathcal{E}}(\mathcal{D})$ then we set $V^* = \{T^* \colon T \in V\}$ (see Proposition 3.1) for the space of all dual operators taken from V. A linear subspace $V \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ is said to be *self-adjoint* (respectively, *unital*) if $V^* = V$ (respectively, $1_{\mathcal{D}} \in V$).

Definition 4.1. Any linear subspace in $C_{\mathcal{E}}(\mathcal{D})$ is called a concrete local operator space on a quantized domain $\mathcal{E}$. A unital self-adjoint subspace in $C^*_{\mathcal{E}}(\mathcal{D})$ is called a local operator system. If $TS \in V$ for all elements T, S of a local operator system V, then we say that V is a local operator algebra.

The completions of the concrete models introduced in Definition 4.1 remain the same concrete models thanks to Proposition 4.1 and its proof. In particular, if $V \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ is a local operator algebra then its completion $\widetilde{V} \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ is a multinormed C^*-algebra called a *local operator C^*-algebra.*

Remark 4.1. Let $V \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ be a local operator system and let $V_\alpha = \{T|H_\alpha\colon T \in V\} \subseteq \mathcal{B}(H_\alpha)$ be the range of the unital $*$-linear mapping $\pi_\alpha : V \to V_\alpha$, $\pi_\alpha(T) = T|H_\alpha$. In particular, V_α is a normed operator system [9, 5.1] on H_α called an operator system associated with the matrix seminorm p_α. If $\alpha, \beta \in \Lambda$, $\alpha \leqslant \beta$, then we have a unital $*$-linear restriction mapping $\pi_{\alpha\beta} : V_\beta \to V_\alpha$ such that $\pi_{\alpha\beta}\pi_\beta = \pi_\alpha$. Thus V is a local operator system subspace of the inverse limit $\varprojlim\{V_\alpha, \pi_{\alpha\beta}\}$ of the operator systems. In particular, the matrix topology of V is just the projective operator space topology [10, Section 6]. Finally, on account of Proposition 4.1, we also deduce that $\widetilde{V} = \varprojlim\{\widetilde{V}_\alpha, \pi_{\alpha\beta}\}$.

A local operator algebra is an example of Op^*-algebra [3,17,19], which is a particular case of a Lassner algebra on H [12, 3.2.3]. It is proved [3] that each F^*-algebra (Fréchet multinormed C^*-algebra) A can be realized as a local operator algebra on some Hilbert space H. The same result is true for the barreled multinormed C^*-algebras [17]. For the general case (see [12, 5.3.46]) we investigate the problem below in Section 7.2 developing a local operator space version of the construction used in [5, Theorem 6] and [13, Theorem 5.1].

4.3. Local positivity and local matrix contraction

Now let us introduce local positivity in a local operator system. Let $V \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ be a local operator system. An element $T \in V$ is called *local hermitian* if $T = T^*$ on a certain subspace H_α, that is, $T|H_\alpha = T^*|H_\alpha = (T|H_\alpha)^*$ in $\mathcal{B}(H_\alpha)$, or $\pi_\alpha(T) = \pi_\alpha(T^*)$ in V_α (see Remark 4.1). In this case, we write $T =_\alpha T^*$. If the latter is true for all α, we say that T is (global) *hermitian*. The set of all local hermitian elements in V is denoted by V_{lh}. An element $T \in V$ is said to be *local positive* if $T \geqslant_\alpha 0$, that is, $T|H_\alpha \geqslant 0$ in $\mathcal{B}(H_\alpha)$ (or $\pi_\alpha(T) \geqslant 0$ in V_α), for some $\alpha \in \Lambda$. Similarly, it is defined a (global) *positive* element in V. The set of all local positive elements in V is denoted by V^+. Evidently, $I_{\mathcal{D}} \in V^+ \subseteq V_{\mathrm{lh}}$. Moreover,

$$V_{\mathrm{lh}} \subseteq V^+ - V^+. \tag{4.1}$$

Indeed, take $T \in V_{\mathrm{lh}}$. Then $T|H_\alpha$ is hermitian in $\mathcal{B}(H_\alpha)$ for some α, and

$$T = \big(T + p^{(1)}_\alpha(T) I_{\mathcal{D}}\big) - p^{(1)}_\alpha(T) I_{\mathcal{D}}.$$

Moreover, $(T + p^{(1)}_\alpha(T) I_{\mathcal{D}})|H_\alpha = T|H_\alpha + p^{(1)}_\alpha(T) I_{H_\alpha} = T|H_\alpha + \|T|H_\alpha\| I_{H_\alpha} \geqslant 0$ in $\mathcal{B}(H_\alpha)$, that is, $T + p^{(1)}_\alpha(T) I_{\mathcal{D}} \in V^+$ and $p^{(1)}_\alpha(T) I_{\mathcal{D}} \in V^+$.

Let $V \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ and $W \subseteq C^*_{\mathcal{K}}(\mathcal{O})$ be local operator systems on the quantized domains $\mathcal{E} = \{H_\alpha\}_{\alpha\in\Lambda}$ and $\mathcal{K} = \{K_\iota\}_{\iota\in\Omega}$ with their union spaces $\mathcal{D}$ and $\mathcal{O}$, respectively. A linear mapping $\varphi : V \to W$ is said to be *local positive* if for each $\iota \in \Omega$ there corresponds $\alpha \in \Lambda$ such that $\varphi(v) \geqslant_\iota 0$ whenever $v \geqslant_\alpha 0$, and $\varphi(v) =_\iota 0$ if $v =_\alpha 0$, $v \in V$. For brevity, we write $\varphi(v) >_\iota 0$ whenever $v >_\alpha 0$. In particular, for each ι the mapping φ can be factored through a positive mapping $\varphi_{\iota\alpha} : V_\alpha \to W_\iota$, $\varphi_{\iota\alpha}\pi_\alpha = \varphi\pi_\iota$, between the relevant (normed) operator systems (see Remark 4.1). Further, a linear mapping $\varphi : V \to W$ is called *local matrix positive* if for each $\iota \in \Omega$

there corresponds $\alpha \in \Lambda$ such that $\varphi^{(n)}(v) \geqslant_\iota 0$ (that is, $\varphi^{(n)}(v)|K_\iota^n \geqslant 0$) whenever $v \geqslant_\alpha 0$ (that is, $v|H_\alpha^n \geqslant 0$), and $\varphi^{(n)}(v) =_\iota 0$ if $v =_\alpha 0$, $v \in M_n(V)$, $n \in \mathbb{N}$. Thus $\varphi^{(n)}(v) >_\iota 0$ whenever $v >_\alpha 0$, $v \in M_n(V)$, $n \in \mathbb{N}$. In this case, the mapping $\varphi_{\iota\alpha} : V_\alpha \to W_\iota$ should be matrix positive. In particular, all $\varphi^{(n)}$ are local positive maps. We say that $\varphi : V \to W$ is a *morphism* if it is a local matrix positive and unital, that is, $\varphi(I_\mathcal{D}) = I_\mathcal{O}$. Finally, if for each $\iota \in \Omega$ there corresponds $\alpha \in \Lambda$ such that $\|\varphi^{(n)}(v)\|_{\mathcal{B}(K_\iota^n)} \leqslant \|v\|_{\mathcal{B}(H_\alpha^n)}$ for all $v \in M_n(V)$, $n \in \mathbb{N}$, then we say that $\varphi : V \to W$ is a *local matrix contraction*. Moreover, if $\Omega = \Lambda$ and $\|\varphi^{(n)}(v)\|_{\mathcal{B}(K_\alpha^n)} = \|v\|_{\mathcal{B}(H_\alpha^n)}$ for all $v \in M_n(V)$, $n \in \mathbb{N}$, $\alpha \in \Lambda$, then φ is called a *local matrix isometry*.

Lemma 4.2. *Let $V \subseteq C^*_\mathcal{E}(\mathcal{D})$ be a local operator system. If $\mathcal{E}$ is a totally ordered family then V_{lh} is a real subspace in V, V^+ is a cone in V, and $V_{\mathrm{lh}} = V^+ - V^+$.*

Proof. Take $T, S \in V_{\mathrm{lh}}$ (respectively, $T, S \in V^+$). According to the definition, $T|H_\alpha = T^*|H_\alpha$ and $S|H_\beta = S^*|H_\beta$ (respectively, $T|H_\alpha \geqslant 0$ and $S|H_\beta \geqslant 0$) for some $\alpha, \beta \in \Lambda$. By assumption, $\alpha \leqslant \beta$ or $\beta \leqslant \alpha$. Assume $\alpha \leqslant \beta$. Then $H_\alpha \subseteq H_\beta$, hence both operators $T|H_\alpha$ and $S|H_\alpha$ are hermitian (respectively, positive) in $\mathcal{B}(H_\alpha)$. Consequently, so is $T - S$ (respectively, $T + S$). Thus $T - S \in V_{\mathrm{lh}}$ (respectively, $T + S \in V^+$). It remains to use (4.1). □

Now we investigate a relationship between local positivity and continuity. We prove that for a unital linear mapping between local operator systems the properties to be local matrix positive and local matrix contractive are equivalent.

Lemma 4.3. *Let V (respectively, W) be either local operator system or unital multinormed C^*-algebra, and let $\varphi : V \to W$ be a local positive mapping. Then $\varphi(v^*) = \varphi(v)^*$ for all $v \in V$. In particular, $\varphi^{(n)}(v^*) = \varphi^{(n)}(v)^*$ for all $v \in M_n(V)$, whenever φ is local matrix positive.*

Proof. Take $v \in V$. One should prove that $\varphi(v^*) = \varphi(v)^*$, that is, $\varphi(v^*) =_\iota \varphi(v)^*$ for all $\iota \in \Omega$. Being φ a local positive mapping, we assert that for each $\iota \in \Omega$ there corresponds $\alpha \in \Lambda$ such that $\varphi(w) \geqslant_\iota 0$ whenever $w \geqslant_\alpha 0$, $w \in V$. Let $\mathrm{Re}(v) = (v + v^*)/2$ and $\mathrm{Im}(v) = (v - v^*)/2i$ be the (global) hermitian elements associated with v, thus $\mathrm{Re}(v)^* =_\gamma \mathrm{Re}(v)$ and $\mathrm{Im}(v)^* =_\gamma \mathrm{Im}(v)$ for all $\gamma \in \Lambda$. If $w = \mathrm{Re}(v)$ (respectively, $w = \mathrm{Im}(v)$) then $w = w_1 - w_2$, where $w_1 = w + p_\alpha^{(1)}(w) I_\mathcal{D}$ and $w_2 = p_\alpha^{(1)}(w) I_\mathcal{D}$. Moreover, $w_k \geqslant_\alpha 0$, $k = 1, 2$. It follows that $\varphi(w_k) \geqslant_\iota 0$, $k = 1, 2$. Therefore $\varphi(w)|K_\iota$ as the difference of positive operators is hermitian, that is, $\varphi(w)^* =_\iota \varphi(w)$. But the latter is true for each ι, whence $\varphi(w)^* = \varphi(w)$. Similar argument can be applied to $w = \mathrm{Im}(v)$.

Thus $\varphi(v)^* = (\varphi(\mathrm{Re}(v)) + i\varphi(\mathrm{Im}(v)))^* = \varphi(\mathrm{Re}(v)) - i\varphi(\mathrm{Im}(v)) = \varphi(v^*)$, $v \in V$. □

Lemma 4.4. *Let V (respectively, W) be either local operator system or unital multinormed C^*-algebra, and let $\varphi : V \to W$ be a local matrix positive linear mapping. Then φ is matrix continuous.*

Proof. Take an index $\iota \in \Omega$. By assumption, there are an index $\alpha \in \Lambda$ and a matrix positive mapping $\varphi_{\iota\alpha} : V_\alpha \to W_\iota$ such that $\varphi_{\iota\alpha}\pi_\alpha = \pi_\iota\varphi$, where V_α (respectively, W_ι) is the normed operator system associated with the seminorm p_α (respectively, q_ι) (see Remark 4.1). Using [9, Lemma 5.5.1], we conclude that $\varphi_{\iota\alpha}$ is matrix bounded, that is, $q_\iota^{(n)}(\varphi^{(n)}(v)) \leqslant C_{\iota\alpha} p_\alpha^{(n)}(v)$ for all $v \in M_n(V)$ and n, where $C_{\iota\alpha} = \|\varphi_{\iota\alpha}(I_{H_\alpha})\| = \|\varphi(I_\mathcal{D})|K_\iota\|$ (in the C^*-algebra case, we put $C_{\iota\alpha} = \|\pi_\iota(\varphi(1_A))\|_{B_\iota}$). Thus φ is matrix continuous. □

Corollary 4.1. *Let V (respectively, W) be either local operator system or unital multinormed C^*-algebra, and let $\varphi : V \to W$ be a unital linear mapping. Then φ is local matrix positive iff φ is a local matrix contraction.*

Proof. If φ is a local matrix positive and $\varphi(I_{\mathcal{D}}) = I_{\mathcal{O}}$, then using Lemma 4.4, we obtain that for each $\iota \in \Omega$ there corresponds $\alpha \in \Lambda$ such that $q_\iota^{(\infty)} \varphi^{(\infty)} \leqslant C_{\iota\alpha} p_\alpha^{(\infty)}$, where $C_{\iota\alpha} = \|\varphi(I_{\mathcal{D}})|K_\iota\| = \|I_{\mathcal{O}}|K_\iota\| = \|I_{K_\iota}\| = 1$. In the C^*-algebra case, we have $C_{\iota\alpha} = \|\pi_\iota \varphi(I_{\mathcal{D}})\|_{B_\iota} = \|\pi_\iota(1_B)\|_{B_\iota} = \|1_{B_\iota}\|_{B_\iota} = 1$ and similarly, $\|\pi_\iota \varphi(1_A)\|_{B_\iota} = 1$. Thus $q_\iota^{(\infty)} \varphi^{(\infty)} \leqslant p_\alpha^{(\infty)}$, which means that φ is a local matrix contraction.

Conversely, assume that φ is a local matrix contraction. According to the definition, for each $\iota \in \Omega$ there corresponds $\alpha \in \Lambda$ such that $q_\iota^{(\infty)} \varphi^{(\infty)} \leqslant p_\alpha^{(\infty)}$. Thus φ can be factored through a matrix contraction $\varphi_{\iota\alpha} : V_\alpha \to W_\iota$ between operator systems (see Remark 4.1). It follows that $\varphi_{\iota\alpha}$ is matrix positive by virtue of [9, Corollary 5.1.2]. Whence φ is local matrix positive. □

5. The Stinespring theorem

In this section we propose a locally convex version of the Stinespring theorem and investigate the decomposition of a local matrix contraction into the contractions and unital contractive $*$-homomorphism.

5.1. The $*$-homomorphism π

First, with each local matrix contractive and local matrix positive mapping from a unital multinormed C^*-algebra A into a local operator system we associate a unital contractive $*$-homomorphism from A into a local operator C^*-algebra.

Everywhere in this section we shall assume that $\{p_\alpha \colon \alpha \in \Lambda\}$ is a (saturated) family of C^*-seminorms of a unital multinormed C^*-algebra A, A_α is the C^*-algebra associated with the C^*-seminorm p_α, $\pi_\alpha : A \to A_\alpha$ is the canonical $*$-homomorphism, $V \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ is a local operator system on a quantized domain $\mathcal{E} = \{H_\iota\}_{\iota \in \Omega}$ in H with its union space $\mathcal{D}$, $\mathfrak{p} = \{P_\iota\}_{\iota \in \Omega}$ is the projection net associated with $\mathcal{E}$, and $\{q_\iota \colon \iota \in \Omega\}$ is a defining family of matrix seminorms on $C^*_{\mathcal{E}}(\mathcal{D})$, where each $q_\iota = (q_\iota^{(n)})_{n \in \mathbb{N}}$ with $q_\iota^{(n)}(v) = \|P_\iota^{\oplus n} T P_\iota^{\oplus n}\|$, $v \in C^*_{\mathcal{E}^n}(\mathcal{D}^n)$. Further, we fix a local matrix contractive and local matrix positive mapping $\varphi : A \to V$ (with respect to the matrix seminorms $\{p_\alpha \colon \alpha \in \Lambda\}$ and $\{q_\iota \colon \iota \in \Omega\}$). Thus for each $\iota \in \Omega$ there corresponds $\alpha, \beta \in \Lambda$ such that $q_\iota^{(\infty)} \varphi^{(\infty)} \leqslant p_\alpha^{(\infty)}$, and $\varphi^{(n)}(a) >_\iota 0$ whenever $a >_\beta 0$ for all $a \in M_n(A)$, $n \in \mathbb{N}$. Take $\gamma \in \Lambda$ with $\gamma \geqslant \alpha$ and $\gamma \geqslant \beta$. Since $p_\alpha^{(\infty)} \leqslant p_\gamma^{(\infty)}$ and the connecting $*$-homomorphism $\pi_{\beta\gamma} : A_\gamma \to A_\alpha$ is matrix positive, we may assume that $\alpha = \beta$. If $V \subseteq \mathcal{B}(H)$ (that is, when $\mathcal{E} = \{H\}$) then we have $\|\varphi^{(n)}(a)\|_{\mathcal{B}(H^n)} \leqslant p_\alpha^{(n)}(a)$, and $\varphi^{(n)}(a) \geqslant 0$ whenever $a \geqslant_\alpha 0$, $a \in M_n(A)$, $n \in \mathbb{N}$. In the latter case, we say that φ is a matrix α-contractive and matrix α-positive. The original inner product in H is denoted by $\langle \cdot | \cdot \rangle$.

Lemma 5.1. *Let $A \otimes \mathcal{D}$ be the algebraic tensor product of the multinormed C^*-algebra A and the union space $\mathcal{D}$ of $\mathcal{E}$, and let $\langle \cdot, \cdot \rangle$ be the sesquilinear form on $A \otimes \mathcal{D}$ determined by the rule $\langle \sum b_j \otimes \eta_j, \sum a_i \otimes \xi_i \rangle = \sum \langle \varphi(a_i^* b_j) \eta_j | \xi_i \rangle$ for all $a_i, b_j \in A$ and $\xi_i, \eta_j \in \mathcal{D}$. Then $\langle \cdot, \cdot \rangle$ is a positive semidefined sesquilinear form on $A \otimes \mathcal{D}$, and therefore it induces a Hilbert space inner product on the quotient space $(A \otimes \mathcal{D})/N$ modulo the subspace $N = \{u \in A \otimes \mathcal{D} \colon \langle u, u \rangle = 0\}$.*

The proof is the same as in the normed case [9, 5.2.1].

Let K be the completion of the pre-Hilbert space $(A \otimes \mathcal{D})/N$ (if $V \subseteq \mathcal{B}(H)$ then K is the completion of $(A \otimes H)/N$) from Lemma 5.1. For each $a \in A$, consider the linear mapping $\pi(a) = L_a \otimes I_{\mathcal{D}} : A \otimes \mathcal{D} \to A \otimes \mathcal{D}$, where $L_a \in L(A)$, $L_a x = ax$, is the left multiplication (by a) operator on A. Take a tensor $u = \sum_{i=1}^{n} a_i \otimes \xi_i \in A \otimes \mathcal{D}$. Being $\{H_\iota\}_{\iota \in \Omega}$ a directed family of subspaces in H, there is an index ι such that $\{\xi_i\} \subseteq H_\iota$, that is, $\xi = (\xi_i) \in H_\iota^n$. For that index ι there corresponds $\alpha \in \Lambda$ with the above confirmed properties of the mapping φ. Note that

$$\langle \pi(a)u, \pi(a)u \rangle = \left\langle \sum aa_i \otimes \xi_i, \sum aa_i \otimes \xi_i \right\rangle = \langle \varphi^{(n)}([a_i^* a^* a a_j])\xi | \xi \rangle$$

and $[a_i^* a^* a a_j] \leqslant_\gamma p_\gamma(a)^2 [a_i^* a_j]$, that is, $\pi_\gamma^{(n)}([a_i^* a^* a a_j]) \leqslant p_\gamma(a)^2 \pi_\gamma^{(n)}([a_i^* a_j])$ for all $\gamma \in \Lambda$. Then $\varphi^{(n)}([a_i^* a^* a a_j]) \leqslant_\iota p_\alpha(a)^2 \varphi^{(n)}([a_i^* a_j])$. By Lemma 5.1,

$$\langle \pi(a)u, \pi(a)u \rangle = \langle \varphi^{(n)}([a_i^* a^* a a_j])\xi | \xi \rangle \leqslant p_\alpha(a)^2 \langle \varphi^{(n)}([a_i^* a_j])\xi | \xi \rangle = p_\alpha(a)^2 \langle u, u \rangle,$$

that is,

$$\langle \pi(a)u, \pi(a)u \rangle \leqslant p_\alpha(a)^2 \langle u, u \rangle \tag{5.1}$$

for all $a \in A$. In particular, $\pi(a)(N) \subseteq N$, therefore it determines a linear mapping on the quotient space $(A \otimes \mathcal{D})/N$ denoted by $\pi(a)$ too. Thus $\pi(a)$ is a densely defined unbounded operator on the Hilbert space K.

Now let $M_\iota = \{(\sum a_j \otimes \xi_j)^\sim (\mathrm{mod}\, N) \colon \{\xi_j\} \subseteq H_\iota\}$ be a subspace in K, and let K_ι be the closure of M_ι in K. The subspace M_ι is just the range of the subspace $A \otimes H_\iota$ via the quotient mapping $(A \otimes \mathcal{D}) \to (A \otimes \mathcal{D})/N$, and it is beyond a doubt it is invariant under the linear mapping $\pi(a)$. Moreover, as follows from (5.1), $\|\pi(a)u\| \leqslant p_\alpha(a)\|u\|$ for all $u \in M_\iota$. Taking into account that $\overline{M_\iota} = K_\iota$, we conclude that the latter inequality is true for all $u \in K_\iota$. Therefore $\pi(a)$ extends up to a bounded linear operator on K_ι, which we denote by $\pi(a)_\iota$. Moreover, $\|\pi(a)_\iota\|_{\mathcal{B}(K_\iota)} \leqslant p_\alpha(a)$, $a \in A$ (confirm again that ι and α are the same indices associated with φ). If $K_\iota \subseteq K_\nu$ then $(\pi(a)_\nu)|K_\iota = \pi(a)_\iota$. Consider the domain $\mathcal{S} = \{K_\iota\}_{\iota \in \Lambda}$ in K and its union space $\mathcal{O}$. We have a well-defined unbounded operator $\pi(a)$ on K such that $\mathrm{dom}(\pi(a)) = \mathcal{O}$ and $\pi(a)|K_\iota = \pi(a)_\iota$ for all ι. Thus $\pi(a)$ leaves invariant each subspace K_ι, and for each $\iota \in \Lambda$ there corresponds $\alpha \in \Omega$ such that $\|\pi(a)|K_\iota\| \leqslant p_\alpha(a)$ for all $a \in A$. Whence $\pi(a) \in C_{\mathcal{S}}(\mathcal{O})$ (see (3.1)). Moreover, the mapping $\pi : A \to C_{\mathcal{S}}(\mathcal{O})$, $a \mapsto \pi(a)$, is a unital homomorphism. If $V \subseteq \mathcal{B}(H)$ then $\mathcal{S} = \{K\}$ and $\|\pi(a)\| \leqslant p_\alpha(a)$ for all $a \in A$, that is, $\pi : A \to \mathcal{B}(K)$, $a \mapsto \pi(a)$, is a unital α-contractive homomorphism, whenever φ is matrix α-contractive.

Lemma 5.2. *The range of π belongs to the multinormed C^*-algebra $C^*_{\mathcal{S}}(\mathcal{O})$ and $\pi : A \to C^*_{\mathcal{S}}(\mathcal{O})$ is a local contractive unital $*$-homomorphism. Moreover, π is a faithful representation whenever* $\ker(\varphi) = \{0\}$.

Proof. Take $a \in A$ and $u^\sim, v^\sim \in M_\iota$ with $u = \sum b_j \otimes \eta_j$, $v = \sum a_i \otimes \xi_i$, where $\{\eta_j, \xi_i\} \subseteq H_\iota$. Then

$$\begin{aligned}\langle \pi(a)u^\sim, v^\sim \rangle &= \langle \pi(a)u, v \rangle = \sum \langle \varphi(a_i^* a b_j)\eta_j | \xi_i \rangle = \sum \langle \varphi((a^* a_i)^* b_j)\eta_j | \xi_i \rangle = \langle u, \pi(a^*)v \rangle \\ &= \langle u^\sim, \pi(a^*)v^\sim \rangle.\end{aligned}$$

By continuity, $\langle \pi(a)x, y\rangle = \langle x, \pi(a^*)y\rangle$ for all $x, y \in K_\iota$. By Proposition 3.1, $\pi(a) \in C^*_{\mathcal{S}}(\mathcal{O})$ and $\pi(a)^* = \pi(a^*)$, for all $a \in A$. Moreover, $\pi(ab) = \pi(a)\pi(b)$ for all $a, b \in A$, therefore $\pi : A \to C^*_{\mathcal{S}}(\mathcal{O})$ is a unital $*$-homomorphism, which is a local contractive mapping by virtue of the construction.

Finally, assume that $\ker(\varphi) = \{0\}$ and let $\pi(a) = 0$ for a certain $a \in A$. Take $u^\sim, v^\sim \in \mathcal{O}$ with $u = 1_A \otimes \eta$, $v = 1_A \otimes \xi$, where $\xi, \eta \in \mathcal{D}$. Then $0 = \langle \pi(a)u^\sim, v^\sim\rangle = \langle a \otimes \eta, 1_A \otimes \xi\rangle = \langle \varphi(a)\eta|\xi\rangle$, in particular, $\langle \varphi(a)\eta|\varphi(a)\eta\rangle = 0$ for all $\eta \in \mathcal{D}$ $(\varphi(a)(\mathcal{D}) \subseteq \mathcal{D})$. Thus $\varphi(a) = 0$, therefore $a = 0$. □

If $V \subseteq \mathcal{B}(H)$ then $C^*_{\mathcal{S}}(\mathcal{O}) = \mathcal{B}(K)$ and $\pi : A \to \mathcal{B}(K)$ is a unital α-contractive $*$-representation, that is, $\|\pi(a)\| \leqslant p_\alpha(a)$ for all $a \in A$.

5.2. *The decomposition theorem*

Now let us prove the main result of the section. Again we assume that A is a unital multi-normed C^*-algebra with its defining family of C^*-seminorms $\{p_\alpha\colon \alpha \in \Lambda\}$ and $\mathcal{E} = \{H_\iota\}_{\iota\in\Omega}$ is a quantized domain in a Hilbert space H with its union space $\mathcal{D}$.

Theorem 5.1. *Let $\varphi : A \to C^*_{\mathcal{E}}(\mathcal{D})$ be a local matrix contractive mapping. If φ is local matrix positive then there are a quantized domain $\mathcal{S} = \{K_\iota\}_{\iota\in\Omega}$ in a Hilbert space K with its union space $\mathcal{O}$, a contraction $T : H \to K$, and unital local contractive $*$-homomorphism $\pi : A \to C^*_{\mathcal{S}}(\mathcal{O})$ such that*

$$T(\mathcal{E}) \subseteq \mathcal{S} \quad \text{and} \quad \varphi(a) \subseteq T^*\pi(a)T$$

for all $a \in A$. Moreover, if $\varphi(1_A) = I_{\mathcal{D}}$ then T is an isometry.

Proof. Let K be the Hilbert space defined above and let π be the representation from Lemma 5.2. Consider a linear mapping $T : \mathcal{D} \to K$ given by the rule $T(\xi) = (1_A \otimes \xi)^\sim \pmod N$. Take $\xi \in \mathcal{D}$. Fix $\iota \in \Omega$ and take $\xi \in H_\iota$. Then $T\xi \in K_\iota \subseteq \mathcal{O}$. Whence $T(H_\iota) \subseteq K_\iota$. By assumption, $q_\iota^{(1)}\varphi \leqslant p_\alpha^{(1)}$ for some α. Therefore,

$$\begin{aligned}
\|T\xi\|^2 = \langle T\xi, T\xi\rangle = \langle 1_A \otimes \xi, 1_A \otimes \xi\rangle = \bigl\langle \varphi(1_A)\xi|\xi\bigr\rangle &\leqslant \bigl\|\varphi(1_A)\xi\bigr\| \|\xi\| \\
&\leqslant \bigl\|\varphi(1_A)|H_\iota\bigr\| \|\xi\|^2 = q_\iota^{(1)}\bigl(\varphi(1_A)\bigr)\|\xi\|^2 \\
&\leqslant p_\alpha^{(1)}(1_A)\|\xi\|^2 = \|\xi\|^2,
\end{aligned}$$

that is, $\|T\xi\| \leqslant \|\xi\|$. If $\varphi(1_A) = I_{\mathcal{D}}$ then $\|T\xi\| = \|\xi\|$. Thus T has unique extension up to a contraction on $\overline{\mathcal{D}} = H$ (respectively, an isometry if $\varphi(1_A) = I_{\mathcal{D}}$).

Finally, take $a \in A$ and $\xi, \eta \in \mathcal{D}$. Then

$$\bigl\langle T^*\pi(a)T\xi|\eta\bigr\rangle = \bigl\langle \pi(a)T\xi, T\eta\bigr\rangle = \bigl\langle (a\otimes\xi)^\sim, (1_A \otimes \eta)^\sim\bigr\rangle = \langle a \otimes \xi, 1_A \otimes \eta\rangle = \bigl\langle \varphi(a)\xi|\eta\bigr\rangle,$$

that is, $\langle (T^*\pi(a)T - \varphi(a))\xi|\eta\rangle = 0$ for all $\xi, \eta \in \mathcal{D}$. Fix $\xi \in \mathcal{D}$. Then $(T^*\pi(a)T - \varphi(a))\xi \in H$ and $\langle (T^*\pi(a)T - \varphi(a))\xi|\eta\rangle = 0$ for all $\eta \in H$. Hence $T^*\pi(a)T\xi = \varphi(a)\xi$. Thus $T^*\pi(a)TP_\iota = \varphi(a)P_\iota$ for all $\iota \in \Omega$. □

If $V = \mathcal{B}(H)$ and φ is unital, then $T : H \to K$, $T(\xi) = (1_A \otimes \xi)^{\sim} (\operatorname{mod} N)$, is an isometry and $T^* \pi(a) T = \varphi(a)$ for all $a \in A$. The following particular cases of Theorem 5.1 present an interest.

Corollary 5.1. *Let $\varphi : A \to C^*_{\mathcal{E}}(\mathcal{D})$ be a unital local matrix positive mapping. Then there are a quantized domain $\mathcal{S}$ in a Hilbert space K containing H and a unital local contractive $*$-homomorphism $\pi : A \to C^*_{\mathcal{S}}(\mathcal{O})$ such that*

$$\mathcal{E} \subseteq \mathcal{S} \quad \textit{and} \quad \varphi(a) \subseteq P_H \pi(a)$$

for all $a \in A$, where $\mathcal{O}$ is the union space of $\mathcal{S}$ and P_H is the projection in $\mathcal{B}(K)$ onto H.

Proof. One should apply Corollary 4.1 and Theorem 5.1. $\square$

Corollary 5.2. *Let $\varphi : A \to M_n$ be a unital matrix α-contractive mapping, that is, $\varphi(1_A) = I_{\mathbb{C}^n}$ and $\|\varphi^{(\infty)}(a)\| \leqslant p_\alpha^{(\infty)}(a)$, $a \in M(A)$. There are a unital α-contractive representation $\pi : A \to \mathcal{B}(K)$ and an isometry $T : \mathbb{C}^n \to K$ such that $T^* \pi(a) T = \varphi(a)$ for all $a \in A$. Moreover, the Hilbert space dimension of K is at most the cardinality* card(A) *of A.*

Proof. If $A = \{0\}$ then the assertion is trivial. Assume $A \neq \{0\}$. Note that φ is matrix α-positive thanks to Corollary 4.1. Assume K is the same as in Theorem 5.1, that is, the norm-completion of the pre-Hilbert space $(A \otimes \mathbb{C}^n)/N$. Thus A^n has the dense range M in K. Take a Hilbert basis $(e_\theta)_{\theta \in \Xi}$ in K. For each $\theta \in \Xi$ take $x_\theta \in M$ such that $\|e_\theta - x_\theta\| \leqslant 2^{-1}$. If $x_\theta = x_\eta$ for some different θ and η from Ξ, then $\sqrt{2} = (\|e_\theta\|^2 + \|e_\eta\|^2)^{1/2} = \|e_\theta - e_\eta\| \leqslant \|e_\theta - x_\theta\| + \|e_\eta - x_\eta\| \leqslant 1$, a contradiction. So, $\operatorname{card}(\Xi) \leqslant \operatorname{card}(M) \leqslant \operatorname{card}(A^n) = \operatorname{card}(A)$.

Finally, $T^* \pi(a) T = \varphi(a)$ for all $a \in A$, and $\pi : A \to \mathcal{B}(K)$ is a unital α-contractive representation due to Theorem 5.1. $\square$

Corollary 5.3. *If there is a unital local matrix isometry $\varphi : A \to C^*_{\mathcal{E}}(\mathcal{D})$ then there is a local isometric $*$-isomorphism $\pi : A \to C^*_{\mathcal{S}}(\mathcal{O})$ for a certain domain $\mathcal{S}$.*

Proof. By assumption, $\Lambda = \Omega$ and $q_\iota^{(\infty)} \varphi^{(\infty)} = p_\iota^{(\infty)}$ for all ι. Being φ a unital local matrix contraction, it is local matrix positive thanks to Corollary 4.1. By Theorem 5.1, there are a quantized domain $\mathcal{S} = \{K_\iota\}_{\iota \in \Lambda}$ with its union space $\mathcal{O}$, a unital local contractive $*$-isomorphism $\pi : A \to C^*_{\mathcal{S}}(\mathcal{O})$, and an isometry $T : H \to K$, $T(H_\iota) \subseteq K_\iota$, $\iota \in \Omega$, such that $\varphi(a) \subseteq T^* \pi(a) T$ for all $a \in A$. As follows from the proof of Theorem 5.1, $\|\pi(a)|K_\iota\| \leqslant p_\iota(a)$, $a \in A$, for each ι. Take unit vectors $x, y \in H_\iota$. Then so are $Tx, Ty \in K_\iota$, and

$$\langle T^* \pi(a) T x, y \rangle = \langle \pi(a) T x, T y \rangle = \langle (\pi(a)|K_\iota) T x, T y \rangle \leqslant \|(\pi(a)|K_\iota) T x\| \|T y\| \leqslant \|\pi(a)|K_\iota\|.$$

It follows that $p_\iota(a) = \|\varphi(a)|H_\iota\| = \|(T^* \pi(a) T)|H_\iota\| \leqslant \|\pi(a)|K_\iota\| \leqslant p_\iota(a)$ for all $a \in A$. Thus $\|\pi(a)|K_\iota\| = p_\iota(a)$, $a \in A$, for each ι, that is, π is a local isometry. $\square$

Corollary 5.4. *Let $A \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ be a closed local operator system on a quantized domain $\mathcal{E}$. If there is an associative multiplication in A which turns it into a unital multinormed C^*-algebra then A is a local operator C^*-algebra up to a local isometric isomorphism.*

Proof. It suffices to apply Corollary 5.3 for the identical embedding $A \hookrightarrow C^*_{\mathcal{E}}(\mathcal{D})$. $\square$

Corollary 5.5. *Let $\varphi : A \to C^*_{\mathcal{E}}(\mathcal{D})$ be a local matrix contractive mapping. If φ is local matrix positive then $\varphi(a)^*\varphi(a) \leqslant \varphi(a^*a)$ on $\mathcal{D}$ for all $a \in A$. If $\varphi(a)^*\varphi(a) = \varphi(a^*a)$ then $\varphi(ba) = \varphi(b)\varphi(a)$ for all $b \in A$.*

Proof. By Theorem 5.1, there are a quantized domain $\mathcal{S} = \{K_\iota\}_{\iota\in\Omega}$ in a Hilbert space K, a unital local contractive $*$-homomorphism $\pi : A \to C^*_{\mathcal{S}}(\mathcal{O})$ and a contraction $T : H \to K$ such that $TP_\iota = Q_\iota TP_\iota$, $\iota \in \Omega$, and $\varphi(a) \subseteq T^*\pi(a)T$ for all $a \in A$, where $\mathfrak{p} = \{P_\iota\}_{\iota\in\Omega}$ and $\mathfrak{q} = \{Q_\iota\}_{\iota\in\Omega}$ are the projection nets associated with the domains $\mathcal{E}$ and $\mathcal{S}$, respectively. Moreover, φ preserves the involutions thanks to Lemma 4.3. Then

$$\begin{aligned}
\varphi(a)^*\varphi(a)P_\iota &= \varphi(a)^*P_\iota\varphi(a)P_\iota = P_\iota\varphi(a)^*P_\iota\varphi(a)P_\iota = \big(\varphi(a)P_\iota\big)^*P_\iota\varphi(a)P_\iota \\
&= \big(T^*\pi(a)TP_\iota\big)^*\varphi(a)P_\iota = \big(T^*\pi(a)Q_\iota TP_\iota\big)^*\varphi(a)P_\iota \\
&= \big((Q_\iota T)^*\pi(a)Q_\iota TP_\iota\big)^*\varphi(a)P_\iota = P_\iota(Q_\iota T)^*\pi(a^*)(Q_\iota T)(Q_\iota T)^*\pi(a)(Q_\iota T)P_\iota \\
&\leqslant P_\iota(Q_\iota T)^*\pi(a^*)Q_\iota\pi(a)(Q_\iota T)P_\iota = P_\iota(Q_\iota T)^*\pi(a^*a)(Q_\iota T)P_\iota \\
&= P_\iota T^*\pi(a^*a)TP_\iota = P_\iota\varphi(a^*a)P_\iota \\
&= \varphi(a^*a)P_\iota
\end{aligned}$$

for all $\iota \in \Lambda$.

Now assume that $\varphi(a)^*\varphi(a) = \varphi(a^*a)$ and consider the mapping $\varphi^{(2)} : M_2(A) \to C^*_{\mathcal{E}^2}(\mathcal{D}^2)$ (see Proposition 3.3 and Remark 3.3), which is local matrix contractive and local matrix positive because of so is φ. Then for each $b \in A$,

$$\varphi^{(2)}\left(\begin{bmatrix} b^* & a \\ a^* & 0 \end{bmatrix}\right)^* \varphi^{(2)}\left(\begin{bmatrix} b^* & a \\ a^* & 0 \end{bmatrix}\right) \leqslant \varphi^{(2)}\left(\begin{bmatrix} b & a \\ a^* & 0 \end{bmatrix}\begin{bmatrix} b^* & a \\ a^* & 0 \end{bmatrix}\right)$$

on $\mathcal{D}^2$. It follows that

$$\begin{bmatrix} T & S \\ S^* & 0 \end{bmatrix} \geqslant 0$$

in $C^*_{\mathcal{E}^2}(\mathcal{D}^2)$, where $T = \varphi(bb^*) - \varphi(b)\varphi(b^*)$ and $S = \varphi(ba) - \varphi(b)\varphi(a)$. Fix $\iota \in \Lambda$. Then $T|H_\iota$ is hermitian. Moreover, one may assume that $T|H_\iota = I_{H_\iota}$. Then $S|H_\iota = 0$. Thus $\varphi(ba) - \varphi(b)\varphi(a) =_\iota 0$. Therefore $\varphi(ba) = \varphi(b)\varphi(a)$. $\square$

Corollary 5.6. *Let $\Phi : C^*_{\mathcal{E}}(\mathcal{D}) \to C^*_{\mathcal{E}}(\mathcal{D})$ be a local matrix contractive and local matrix positive projection. Then*

$$\Phi\big(\Phi(a)b\big) = \Phi\big(\Phi(a)\Phi(b)\big) = \Phi\big(a\Phi(b)\big)$$

for all $a, b \in C^*_{\mathcal{E}}(\mathcal{D})$.

Proof. Assume that $\mathcal{E} = \{H_\alpha\}_{\alpha\in\Lambda}$ and $\mathfrak{p} = \{P_\alpha\}_{\alpha\in\Lambda}$ is the projection net in $\mathcal{B}(H)$ associated with $\mathcal{E}$. One suffices to prove the equalities for the hermitian elements $a, b \in C^*_{\mathcal{E}}(\mathcal{D})$. We are using the similar argument as in [9, Lemma 6.1.2]. Evidently,

$$d = \begin{bmatrix} 0 & \Phi(a) \\ \Phi(a) & b \end{bmatrix}$$

is hermitian in $M_2(C^*_{\mathcal{E}}(\mathcal{D})) = C^*_{\mathcal{E}^2}(\mathcal{D}^2)$ (see Proposition 3.3). Moreover, $\Phi^{(2)} : C^*_{\mathcal{E}^2}(\mathcal{D}^2) \to C^*_{\mathcal{E}^2}(\mathcal{D}^2)$ is local matrix contractive and local matrix positive too. Then $\Phi^{(2)}(d)$ is hermitian (see Lemma 4.3) and $\Phi^{(2)}(d)^2 \leqslant \Phi^{(2)}(d^2)$ on $\mathcal{D}^2$ thanks to Corollary 5.5. Moreover, $\Phi^{(2)}(\Phi^{(2)}(d)^2) \leqslant \Phi^{(2)}(\Phi^{(2)}(d^2))$ on $\mathcal{D}^2$, for Φ is local matrix positive. It follows that

$$D = \begin{bmatrix} 0 & T \\ T^* & S \end{bmatrix} \geqslant 0 \quad \text{in } C^*_{\mathcal{E}^2}\big(\mathcal{D}^2\big),$$

where $T = \Phi(\Phi(a)b) - \Phi(\Phi(a)\Phi(b))$ and $S = \Phi(\Phi(a)^2 + b^2) - \Phi(\Phi(a)^2 + \Phi(b)^2)$. Note that

$$D|H_\alpha^2 = \begin{bmatrix} 0 & T|H_\alpha \\ (T|H_\alpha)^* & S|H_\alpha \end{bmatrix} \geqslant 0 \quad \text{in } \mathcal{B}\big(H_\alpha^2\big).$$

Whence $T|H_\alpha = 0$ for all α, that is, $\Phi(\Phi(a)b) = \Phi(\Phi(a)\Phi(b))$ and $\Phi(b\Phi(a)) = \Phi(\Phi(b)\Phi(a))$, which in turn implies that $\Phi(\Phi(a)b) = \Phi(\Phi(a)\Phi(b)) = \Phi(a\Phi(b))$. The rest is clear. $\square$

6. The sup-formulas

In this section we describe a continuous matrix seminorm on a local operator space in terms of the matrix duality. The result is based upon the Bipolar theorem which we formulate below.

6.1. *The matrix seminorm case*

Let V and W be a (Hausdorff) polynormed spaces. These spaces are said to be in duality if there is a pairing $\langle\cdot,\cdot\rangle : V \times W \to \mathbb{C}$ such that all continuous functionals on V are given by the elements of W, and vice versa. We briefly say that (V, W) is a dual pair. Thus both topologies on V and W are compatible with the distinguished duality $\langle\cdot,\cdot\rangle$. For instance, the spaces V and V' are in the canonical duality $\langle x, f\rangle = f(x)$, where $V' = \mathcal{C}(V, \mathbb{C})$ is the space of all continuous linear functionals on V. The given pairing between V and W determines a *matrix pairing*

$$\langle\langle\cdot,\cdot\rangle\rangle : \mathbb{M}_m(V) \times \mathbb{M}_n(W) \to M_{mn}, \qquad \langle\langle v, w\rangle\rangle = \big[\langle v_{ij}, w_{st}\rangle\big] = w^{(m)}(v) = v^{(n)}(w),$$

where $v = [v_{ij}] \in \mathbb{M}_m(V)$, $w = [w_{st}] \in \mathbb{M}_n(W)$, which are identified with the canonical linear mappings

$$v : W \to M_m, \quad v(y) = \big[\langle v_{ij}, y\rangle\big], \quad \text{and} \quad w : V \to M_n, \quad w(x) = \big[\langle x, w_{st}\rangle\big],$$

respectively. Each $\mathbb{M}_m(V)$ (respectively, $\mathbb{M}_n(W)$) endowed with the polynormed topology induced from V^{n^2} (respectively, W^{n^2}), is denoted by $M_m(V)$ (respectively, $M_n(W)$). In particular, the polynormed spaces $M_n(V)$ and $M_n(W)$ are equipped with the relevant weak topologies $\sigma(M_n(V), M_n(W))$ and $\sigma(M_n(W), M_n(V))$ determined by the scalar pairing

$$\langle\cdot,\cdot\rangle : M_n(V) \times M_n(W) \to \mathbb{C}, \quad \langle v, w\rangle = \sum_{i,j} \langle v_{ij}, w_{ij}\rangle.$$

Moreover, the bilinear mapping $\langle\langle\cdot,\cdot\rangle\rangle : V \times M_n(W) \to M_n$ determines all continuous linear mappings $\varphi : V \to M_n$, that is, $M_n(W) = \mathcal{C}(V, M_n)$.

Given a matrix set $\mathfrak{B}$ in $M(V)$ let us introduce the *absolute operator polar* $\mathfrak{B}^{\odot}$ *in* $M(W)$ to be a matrix set $(\mathfrak{b}_n^{\odot})$ defined as

$$\mathfrak{b}_n^{\odot} = \bigl\{w \in M_n(W):\ \bigl\|\langle\langle v, w\rangle\rangle\bigr\| \leqslant 1,\ v \in \mathfrak{b}_s,\ s \in \mathbb{N}\bigr\}.$$

Similarly, one can define the absolute operator polar $\mathfrak{M}^{\odot} \subseteq M(V)$ for a matrix set $\mathfrak{M} = (\mathfrak{m}_n)$ in $M(W)$. A matrix set $\mathfrak{B}$ in $M(V)$ is said to be *weakly closed* if each $\mathfrak{b}_n$ is $\sigma(M_n(V), M_n(W))$-closed in $M_n(V)$. Note that $\mathfrak{B}^{\odot}$ is an absolutely matrix convex and weakly closed set in $M(W)$ [10]. It can be proved that $\mathfrak{b}_1^{\odot}$ coincides with the classical absolute polar of $\mathfrak{b}_1$ in W, that is, $\mathfrak{b}_1^{\odot} = \mathfrak{b}_1^{\circ} = \{w \in W:\ |\langle v, w\rangle| \leqslant 1,\ v \in \mathfrak{b}_1\}$.

The classical Bipolar theorem asserts that the double absolute polar $S^{\circ\circ}$ of a subset $S \subseteq V$ is the smallest weakly closed absolutely convex set containing S. The operator version of this result was proved in [10] by Effros and Webster.

Theorem 6.1. *Let (V, W) be a dual pair and let $\mathfrak{B}$ be a matrix set in $M(V)$. Then $\mathfrak{B}^{\odot\odot}$ is the smallest weakly closed absolutely matrix convex set containing $\mathfrak{B}$. In particular, $\mathfrak{B} = \mathfrak{B}^{\odot\odot}$ for a weakly closed absolutely matrix convex matrix set $\mathfrak{B}$ in $M(V)$.*

Let (V, W) be a dual pair and let $\mathfrak{B} = (\mathfrak{b}_n)$ be a matrix set in $M(V)$. Let us introduce a family $q_{\mathfrak{B}} = (q_{\mathfrak{B}}^{(n)})_{n\in\mathbb{N}}$ of gauges on $M(W)$ defined as

$$q_{\mathfrak{B}}^{(n)}(w) = \sup\bigl\{\bigl\|\langle\langle v, w\rangle\rangle\bigr\|:\ v \in \mathfrak{b}_r,\ r \in \mathbb{N}\bigr\}, \quad w \in M_n(W),\ n \in \mathbb{N}.$$

It can easily be verified that $q_{\mathfrak{B}}$ is a matrix gauge on W called *a dual gauge of* $\mathfrak{B}$. If $\mathfrak{B}$ is the unit set of a matrix gauge $p = (p^{(n)})_{n\in\mathbb{N}}$ on V then $q_{\mathfrak{B}}$ is called a *dual gauge of* p and it is denoted by $p^{\odot} = (p_n^{\odot})_{n\in\mathbb{N}}$. Thus

$$p_n^{\odot}(w) = \sup\bigl\{\bigl\|w^{(r)}(v)\bigr\|:\ v \in M_r(V),\ p^{(r)}(v) \leqslant 1,\ r \in \mathbb{N}\bigr\}$$

for all $w \in M_n(W)$, $n \in \mathbb{N}$. We also introduce the subset $\mathcal{CB}_p(V, M_n) \subseteq M_n(W)$ of all matrix p-contractive linear maps $w : V \to M_n$, that is, $p_n^{\odot}(w) \leqslant 1$.

Lemma 6.1. *Let $\mathfrak{B} = (\mathfrak{b}_n)$ be a matrix set in $M(V)$ and let $q_{\mathfrak{B}} = (q_{\mathfrak{B}}^{(n)})_{n\in\mathbb{N}}$ be its dual gauge. Then $q_{\mathfrak{B}}$ is the Minkowski functional of the absolute operator polar $\mathfrak{B}^{\odot} = (\mathfrak{b}_n^{\odot})$ in $M(W)$. In particular, $\mathfrak{b}_n^{\odot}$ is the unit set of the dual gauge $q_{\mathfrak{B}}^{(n)}$, $n \in \mathbb{N}$.*

Proof. For a while, we denote the matrix gauge of $\mathfrak{B}^{\odot}$ by $\gamma = (\gamma^{(n)})_{n\in\mathbb{N}}$. Let us prove that $\gamma = q_{\mathfrak{B}}$. First note that the set $\mathfrak{b}_n^{\odot}$ is $\sigma(M_n(W), M_n(V))$-closed by its very definition. Moreover,

$\mathfrak{b}_n^{\odot}$ is absolutely convex, for $\mathfrak{B}^{\odot}$ is an absolutely matrix convex set. It follows that $\mathfrak{b}_n^{\odot} = \{\gamma^{(n)} \leqslant 1\}$ (if $\gamma^{(n)}(w) = 1$ then $w_\varepsilon = (1+\varepsilon)^{-1} w \in \mathfrak{b}_n^{\odot}$ for any $\varepsilon > 0$, and $w = \lim_{\varepsilon \to 0} w_\varepsilon \in \mathfrak{b}_n^{\odot}$).

Further, take $t > 0$ with $t^{-1} w \in \mathfrak{b}_n^{\odot}$. Then $\| \langle\langle v, t^{-1} w \rangle\rangle \| \leqslant 1$ for all $v \in \mathfrak{b}_r$, $r \in \mathbb{N}$. Thereby $q_{\mathfrak{B}}^{(n)}(w) \leqslant t$. So, if $\gamma^{(n)}(w) < \infty$ then $q_{\mathfrak{B}}^{(n)}(w) \leqslant \gamma^{(n)}(w)$. Moreover, if $q_{\mathfrak{B}}^{(n)}(w) = 0$ then $\| \langle\langle v, \mu w \rangle\rangle \| = 0$ for all $\mu > 0$ and $v \in \mathfrak{b}_r$, $r \in \mathbb{N}$, that is, $\mu w \in \mathfrak{b}_n^{\odot}$ for all $\mu > 0$. Thus $\gamma^{(n)}(w) = 0$. Further, if $q_{\mathfrak{B}}^{(n)}(w) > 0$ then $\| \langle\langle v, (q_{\mathfrak{B}}^{(n)}(w))^{-1} w \rangle\rangle \| \leqslant 1$ for all $v \in \mathfrak{b}_r$, $r \in \mathbb{N}$. Whence $(q_{\mathfrak{B}}^{(n)}(w))^{-1} w \in \mathfrak{b}_n^{\odot}$, so $\gamma^{(n)}(w) \leqslant q_{\mathfrak{B}}^{(n)}(w)$.

Finally, if $q_{\mathfrak{B}}^{(n)}(w) < \infty$ then $\| \langle\langle v, w \rangle\rangle \| \leqslant t$ for all $v \in \mathfrak{b}_r$, $r \in \mathbb{N}$, and for a certain $t > 0$. It follows that $w \in t\mathfrak{b}_n^{\odot}$, that is, $\gamma^{(n)}(w) < \infty$. Thus $q_{\mathfrak{B}}^{(n)} = \gamma^{(n)}$ for all n. □

Proposition 6.1. *Let (V, W) be a dual pair and let p be a matrix gauge on V with its weakly closed unit set. Then $p = p^{\odot\odot}$ and*

$$p^{(n)}(v) = \sup\bigl\{ \bigl\| \langle\langle v, w \rangle\rangle \bigr\| \colon w \in \mathcal{CB}_p(V, M_r),\ r \in \mathbb{N} \bigr\}, \quad v \in M_n(V).$$

In particular, if p and q are matrix gauges on V with their weakly closed unit sets then

$$p \preccurlyeq q \quad \textit{iff} \quad \mathcal{CB}_p(V, M_r) \subseteq \mathcal{CB}_q(V, M_r), \quad r \in \mathbb{N}.$$

Proof. Let $\mathfrak{b}_n$ be the unit set of the seminorm $p^{(n)}$, $n \in \mathbb{N}$. Then $\mathfrak{B} = (\mathfrak{b}_n)$ is a weakly closed absolutely matrix convex set in $M(V)$. By Bipolar theorem 6.1, $\mathfrak{B} = \mathfrak{B}^{\odot\odot}$. By Lemma 6.1, $\mathfrak{B}^{\odot}$ is the collection of unit sets of the dual gauge $p^{\odot}$. On the same ground, $\mathfrak{B}^{\odot\odot}$ is the collection of unit sets of the gauge $p^{\odot\odot}$. Since $\mathfrak{B} = \mathfrak{B}^{\odot\odot}$, we conclude that $p = p^{\odot\odot}$. In particular, using the latter equality, we derive that

$$\begin{aligned} p^{(n)}(v) &= \sup\bigl\{ \bigl\| v^{(r)}(w) \bigr\| \colon w \in \mathfrak{b}_r^{\odot},\ r \in \mathbb{N} \bigr\} = \sup\bigl\{ \bigl\| \langle\langle v, w \rangle\rangle \bigr\| \colon w \in M_r(W),\ p_r^{\odot}(w) \leqslant 1,\ r \in \mathbb{N} \bigr\} \\ &= \sup\bigl\{ \bigl\| \langle\langle v, w \rangle\rangle \bigr\| \colon w \in \mathcal{CB}_p(V, M_r),\ r \in \mathbb{N} \bigr\}. \end{aligned}$$

Finally, let p and q be matrix gauges on V with their weakly closed unit sets $\mathfrak{B} = (\mathfrak{b}_n)$ and $\mathfrak{M} = (\mathfrak{m}_n)$, respectively. If $p \preccurlyeq q$ then $\mathfrak{M} \subseteq \mathfrak{B}$. Moreover, if $w \in M_n(W)$ then

$$\begin{aligned} q_n^{\odot}(w) &= \sup\bigl\{ \bigl\| \langle\langle v, w \rangle\rangle \bigr\| \colon v \in \mathfrak{m}_r,\ r \in \mathbb{N} \bigr\} \\ &\leqslant \sup\bigl\{ \bigl\| \langle\langle v, w \rangle\rangle \bigr\| \colon v \in \mathfrak{b}_r,\ r \in \mathbb{N} \bigr\} = p_n^{\odot}(w). \end{aligned}$$

It follows that $\mathcal{CB}_p(V, M_n) \subseteq \mathcal{CB}_q(V, M_n)$. Conversely, if the latter takes place then

$$\begin{aligned} p^{(n)}(v) &= \sup\bigl\{ \bigl\| \langle\langle v, w \rangle\rangle \bigr\| \colon w \in \mathcal{CB}_p(V, M_r),\ r \in \mathbb{N} \bigr\} \\ &\leqslant \sup\bigl\{ \bigl\| \langle\langle v, w \rangle\rangle \bigr\| \colon w \in \mathcal{CB}_q(V, M_r),\ r \in \mathbb{N} \bigr\} = q^{(n)}(v) \end{aligned}$$

for all $v \in M_n(V)$. Whence $p \preccurlyeq q$. □

6.2. The C^*-seminorm case

Let A be a unital multinormed C^*-algebra with a (saturated) family of C^*-seminorms $\{p_\alpha\colon \alpha\in\Lambda\}$. Recall that a $*$-representation $\pi: A\to\mathcal{B}(H_\pi)$ is called α-contractive if $\|\pi(a)\|\leqslant p_\alpha(a)$ for all $a\in A$. Evidently, α-contractive $*$-representations are exactly those ones which can be factored through the C^*-algebra A_α associated with the C^*-seminorm p_α. Let us introduce the class $\mathfrak{s}_\alpha$ of those α-contractive $*$-representations $\pi: A\to\mathcal{B}(H_\pi)$ such that the Hilbert space dimension of H_π is at most the cardinality of A. Actually, $\mathfrak{s}_\alpha$ can be embedded into the set of all linear maps from A into $\mathcal{B}(\ell_2(A))$, therefore it is a set. Without that assumption on the Hilbert space dimensions one cannot declare that the class of all Hilbert space representations is a set. Note that $\mathfrak{s}_\alpha\subseteq\mathfrak{s}_\beta$ whenever $p_\alpha\leqslant p_\beta$, $\alpha,\beta\in\Lambda$. Put $\mathfrak{s}=\bigcup_{\alpha\in\Lambda}\mathfrak{s}_\alpha$, which is a set too. The following sup-formula is a C^*-version of one stated in Proposition 6.1.

Proposition 6.2. *The equality $p_\alpha(a)=\sup\{\|\pi(a)\|_{\mathcal{B}(H_\pi)}\colon \pi\in\mathfrak{s}_\alpha\}$ is true for all $a\in A$, $\alpha\in\Lambda$. In particular, $\mathfrak{s}_\alpha$ is a non-empty set.*

Proof. Take $a\in A$, and let $\bar a$ be the range of a in the C^*-algebra A_α generated by the C^*-seminorm p_α, via the canonical mapping $\pi_\alpha: A\to A_\alpha$. If $\|\bar a\|_\alpha$ is the norm of $\bar a$ then $\|\bar a\|_\alpha=p_\alpha(a)$. By Hahn–Banach theorem, there is an α-contractive functional $\varphi: A\to\mathbb{C}$ (that is, $|\varphi(c)|\leqslant p_\alpha(c)$, $c\in A$) such that $\varphi(a)=p_\alpha(a)$. Using [10, Lemma 5.2], we conclude that $\varphi: A\to\mathbb{C}$ is a matrix α-contractive mapping, that is, $\|\varphi^{(\infty)}(c)\|\leqslant p_\alpha^{(\infty)}(c)$, $c\in M(A)$. It follows that there is a matrix contraction $\bar\varphi: A_\alpha\to\mathbb{C}$ such that $\varphi=\bar\varphi\cdot\pi_\alpha$. Using Paulsen's 2×2 'off-diagonalization technique' (see [9, Theorem 5.3.2]), infer that there are unital contractive functionals $\overline{\psi}_1,\overline{\psi}_2\in A_\alpha^*$ such that the mapping

$$\overline{\Phi}=\begin{bmatrix}\overline{\psi}_1 & \bar\varphi\\ \bar\varphi^* & \overline{\psi}_2\end{bmatrix}: M_2(A_\alpha)\to M_2$$

is a morphism. On account of Corollary 4.1, $\overline{\Phi}$ is a unital matrix contraction. It follows that $\Phi=\overline{\Phi}\cdot\pi_\alpha^{(2)}: M_2(A)\to M_2$ is a unital matrix α-contractive mapping. Furthermore,

$$\Phi(c)=\begin{bmatrix}\psi_1(c) & \varphi(c)\\ \varphi^*(c) & \psi_2(c)\end{bmatrix}$$

with $\psi_i=\overline{\psi}_i\pi_\alpha$, $i=1,2$. Using Corollary 5.2, we obtain that $\Phi(c)=V^*\pi_0(c)V$, $c\in M_2(A)$, for some unital α-contractive representation $\pi_0: M_2(A)\to\mathcal{B}(K)$ and an isometry $V:\mathbb{C}^2\to K$. Moreover, the Hilbert space dimension of K is at most $\operatorname{card}(M_2(A))=\operatorname{card}(A)$. Put $\pi: A\to\mathcal{B}(K)$, $\pi(c)=\pi_0(c\oplus c)$. Then $\|\pi(c)\|\leqslant p_\alpha^{(2)}(c\oplus c)\leqslant p_\alpha(c)$ (M1) for all $c\in A$, that is, π is a unital α-contractive $*$-representation. Thereby $\pi\in\mathfrak{s}_\alpha$. Further,

$$\begin{aligned}\Phi\begin{bmatrix}0 & c\\ 0 & 0\end{bmatrix}&=\begin{bmatrix}0 & \varphi(c)\\ 0 & 0\end{bmatrix}=V^*\pi_0\left(\begin{bmatrix}0 & c\\ 0 & 0\end{bmatrix}\right)V=V^*\pi_0\left(\begin{bmatrix}c & 0\\ 0 & c\end{bmatrix}\begin{bmatrix}0 & 1_A\\ 0 & 0\end{bmatrix}\right)V\\ &=V^*\pi(c)\pi_0\begin{bmatrix}0 & 1_A\\ 0 & 0\end{bmatrix}V.\end{aligned}$$

Take the vectors $x, y \in K$ determined as

$$x = \pi_0 \begin{bmatrix} 0 & 1_A \\ 0 & 0 \end{bmatrix} V \begin{bmatrix} 0 \\ 1 \end{bmatrix} (1) \quad \text{and} \quad \langle \cdot, y \rangle = [\,1 \;\; 0\,] V^*,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in K. Then

$$\begin{aligned} \bigl\langle \pi(c)x, y \bigr\rangle &= [\,1 \;\; 0\,] V^* \bigl(\pi(c)x\bigr) = [\,1 \;\; 0\,] V^* \pi(c) \pi_0 \begin{bmatrix} 0 & 1_A \\ 0 & 0 \end{bmatrix} V \begin{bmatrix} 0 \\ 1 \end{bmatrix} (1) \\ &= [\,1 \;\; 0\,] \begin{bmatrix} 0 & \varphi(c) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (1) \\ &= \varphi(c). \end{aligned}$$

Moreover,

$$\|x\| \leqslant \left\| \pi_0 \begin{bmatrix} 0 & 1_A \\ 0 & 0 \end{bmatrix} \right\| \leqslant p_\alpha^{(2)} \begin{bmatrix} 0 & 1_A \\ 0 & 0 \end{bmatrix} = \left\| \begin{bmatrix} 0 & 1_{A_\alpha} \\ 0 & 0 \end{bmatrix} \right\|_{M_2(A_\alpha)} \leqslant 1,$$

and $\|y\| = \|[\,1 \;\; 0\,] V^*\| \leqslant 1$.

Finally, $p_\alpha(a) = \varphi(a) = \langle \pi(a)x, y \rangle \leqslant \|\pi(a)x\|_K \|y\|_K \leqslant \|\pi(a)\|_{\mathcal{B}(K)} \leqslant p_\alpha(a)$, that is, $\|\pi(a)\|_{\mathcal{B}(K)} = p_\alpha(a)$ with $\pi \in \mathfrak{s}_\alpha$. □

7. The representation theorems

In this section we prove the representation theorems for local operator spaces and multinormed C^*-algebras.

7.1. The representation theorem for local operator spaces

We start with the representation theorem for local operator spaces.

Theorem 7.1. *Let V be a local operator space. There is a topological matrix isomorphism $\varphi : V \to C_{\mathcal{E}}(\mathcal{D})$ of V into the Arens–Michael algebra $C_{\mathcal{E}}(\mathcal{D})$ of all noncommutative continuous functions on a certain quantized domain $\mathcal{E}$ with its union space $\mathcal{D}$. Thus V is a concrete local operator space up to a topological matrix isomorphism.*

Proof. Consider the dual pair (V, V'). Since any unit set $\mathfrak{B} = (\mathfrak{b}_n)$ of a continuous matrix seminorm on V is closed and absolutely matrix convex, it follows that each $\mathfrak{b}_n$ being a convex and closed set turns out to be $\sigma(M_n(V), M_n(V'))$-closed by virtue of Mazur's theorem [14, 10.4.6]. Therefore we may use the sup-formula from Proposition 6.1 with respect to any continuous matrix seminorm on V.

Take a defining (saturated) family $\{p_\alpha \colon \alpha \in \Lambda\}$ of matrix seminorms on V. We complete this family by adding all positive rational multipliers cp_α, $(\alpha, c) \in \Lambda \times \mathbb{Q}_+$, and put $(\alpha, c) \preccurlyeq (\beta, d)$ if $cp_\alpha \preccurlyeq dp_\beta$. For instance, the latter takes place whenever $\alpha \leqslant \beta$ and $c \leqslant d$. The family $\{cp_\alpha \colon (\alpha, c) \in \Lambda \times \mathbb{Q}_+\}$ is saturated too. Put $\mathfrak{s}_{(\alpha,c)}^{(r)} = \mathcal{CB}_{cp_\alpha}(V, M_r)$ and $\mathfrak{s}_{(\alpha,c)} = \bigcup_{r \in \mathbb{N}} \mathfrak{s}_{(\alpha,c)}^{(r)}$, where $(\alpha, c) \in \Lambda \times \mathbb{Q}_+$. By Proposition 6.1, $\mathfrak{s}_{(\alpha,c)} \subseteq \mathfrak{s}_{(\beta,d)}$ whenever $(\alpha, c) \preccurlyeq (\beta, d)$. It follows that $\{\mathfrak{s}_{(\alpha,c)}\colon (\alpha, c) \in \Lambda \times \mathbb{Q}_+\}$ is a directed family of sets. Put $\mathfrak{s} = \bigcup_{(\alpha,c)\in\Lambda\times\mathbb{Q}_+} \mathfrak{s}_{(\alpha,c)}$. Take $(\alpha, c) \in \Lambda \times \mathbb{Q}_+$. Let us introduce (as in [9, Theorem 2.3.5]) the Hilbert spaces $H_{(\alpha,c)} = \bigoplus_{w\in\mathfrak{s}_{(\alpha,c)}} \mathbb{C}^{n(w)}$ (the Hilbert space direct sum), where $n(w) = n$ whenever $w \in \mathfrak{s}^{(n)}_{(\alpha,c)}$, $(\alpha, c) \in \Lambda \times \mathbb{Q}_+$. Obviously, $H_{(\alpha,c)}$ is a closed subspace in $H_{(\beta,d)}$ if $(\alpha, c) \preccurlyeq (\beta, d)$. Moreover, all $H_{(\alpha,c)}$ are closed subspaces of the Hilbert space $H = \bigoplus_{w\in\mathfrak{s}} \mathbb{C}^{n(w)}$, where $n(w) = n$ whenever $w \in \mathfrak{s}^{(n)}_{(\alpha,c)}$ for some (α, c) and n. So, we have an upward filtered family $\mathcal{E} = \{H_{(\alpha,c)}\}_{(\alpha,c)\in\Lambda\times\mathbb{Q}_+}$ of closed subspaces in H. The relevant projections in $\mathcal{B}(H)$ onto the subspaces $H_{(\alpha,c)}$ are denoted by $P_{(\alpha,c)}$, respectively, and let $\mathfrak{p} = \{P_{(\alpha,c)}\}$. Since $\lim \mathfrak{p} = 1_H$, it follows that the set $\mathcal{D} = \bigcup \mathcal{E}$ is a dense subspace in H (see Corollary 3.1). Hence $\mathcal{E}$ is a quantized domain in H with its union space $\mathcal{D}$. Further, let us introduce a linear mapping $\Phi_{(\alpha,c)} : V \to \mathcal{B}(H_{(\alpha,c)})$, $\Phi_{(\alpha,c)}(v) = (w(v))_{w\in\mathfrak{s}_{(\alpha,c)}}$. Note that $\Phi^{(n)}_{(\alpha,c)} : M_n(V) \to \mathcal{B}(H^n_{(\alpha,c)})$ is acting by the rule $\Phi^{(n)}_{(\alpha,c)}(v) = (w^{(n)}(v))_{w\in\mathfrak{s}_{(\alpha,c)}}$, $v \in M_n(V)$. Hence

$$\left\|\Phi^{(n)}_{(\alpha,c)}(v)\right\|_{\mathcal{B}(H^n_{(\alpha,c)})} = \sup_{w\in\mathfrak{s}_{(\alpha,c)}} \left\|w^{(n)}(v)\right\| = \sup\left\{\left\|\langle\langle v, w\rangle\rangle\right\|\colon w \in \mathfrak{s}_{cp_\alpha}\right\} = cp^{(n)}_\alpha(v)$$

by virtue of Proposition 6.1. Thus

$$\left\|\Phi^{(n)}_{(\alpha,c)}(v)\right\|_{\mathcal{B}(H^n_{(\alpha,c)})} = cp^{(n)}_\alpha(v) \tag{7.1}$$

for all $v \in M_n(V)$. Consider the linear mapping

$$\Phi : V \to C_{\mathcal{E}}(\mathcal{D}), \quad \Phi(v) = \Phi_{(\alpha,c)}(v) \text{ on } H_{(\alpha,c)}.$$

Since $\Phi_{(\beta,d)}(v)|H_{(\alpha,c)} = \Phi_{(\alpha,c)}(v)$ if $(\alpha, c) \preccurlyeq (\beta, d)$, it follows that Φ is well-defined. Furthermore, if $\Phi(v) = 0$ then $\Phi_{(\alpha,c)}(v) = 0$ for all (α, c). Hence $p^{(1)}_\alpha(v) = 0$ for all α, and therefore $v = 0$. Thus Φ is a linear mapping with $\ker(\Phi) = \{0\}$.

Now let us prove that the quantized domain $\mathcal{E}$ can be replaced by one which does not depend upon the defining family $\{p_\alpha\colon \alpha \in \Lambda\}$ of matrix seminorms on V. Indeed, take another saturated family $\{q_\iota\colon \iota \in \Omega\}$ of matrix seminorms on V. Thereby we have another quantized domain $\mathcal{K} = \{H_{(\iota,c)}\}_{(\iota,c)\in\Omega\times\mathbb{Q}_+}$ in H. For each $\iota \in \Omega$ relates p_α and $t \in \mathbb{Q}_+$ with $q_\iota \preccurlyeq tp_\alpha$, and vice versa. By Proposition 6.1, $\mathfrak{s}_{(\iota,c)} \subseteq \mathfrak{s}_{(\alpha,ct)}$ for all $c \in \mathbb{Q}_+$, hence $H_{(\iota,c)} \subseteq H_{(\alpha,ct)}$, and vice versa. Moreover, $\Phi_{(\alpha,ct)}(v)|H_{(\iota,c)} = \Phi_{(\iota,c)}(v)$. Consequently, $\mathcal{K} \sim \mathcal{E}$. In particular, $\bigcup \mathcal{K} = \mathcal{D}$. Moreover, $\Phi(V) \subseteq C_{\mathcal{E}}(\mathcal{D}) \cap C_{\mathcal{K}}(\mathcal{D}) = C_{\mathcal{E}'}(\mathcal{D})$ (see Proposition 3.2), where $\mathcal{E}' = \mathcal{E} \vee \mathcal{K}$. One may reinforce the domain $\mathcal{E}$ by adding all equivalent domains $\mathcal{K}$ associated with the families of matrix seminorms on V which are equivalent to $\{p_\alpha\colon \alpha \in \Lambda\}$. On the ground of Proposition 3.2, we conclude that $\Phi(V) \subseteq \bigcap_{\mathcal{K}} C_{\mathcal{K}}(\mathcal{D}) = C_{\vee\mathcal{K}}(\mathcal{D})$ and the quantized domain $\bigvee \mathcal{K}$ depends upon just the matrix topology on V.

Thus $\Phi(V)$ is a concrete local operator space on $\mathcal{E}$ and Φ is a topological matrix isomorphism (see (7.1)) of V onto $\Phi(V)$. $\square$

Remark 7.1. In the just proposed proof one may replace the family $\{cp_\alpha\colon (\alpha, c) \in \Lambda \times \mathbb{Q}_+\}$ by its any dominating subset $\{c_\alpha p_\alpha\colon \alpha \in \Lambda\}$. The latter means that for each $c \in \mathbb{Q}_+$ and $\alpha \in \Lambda$ there corresponds $\beta \in \Lambda$ with $cp_\alpha \preccurlyeq c_\beta p_\beta$.

Remark 7.2. Assume that V has a dominating family $\{c_\alpha p_\alpha\colon \alpha \in \Lambda\}$ of matrix seminorms (see Remark 7.1), which is totally ordered, that is, for a couple of indices $\alpha, \beta \in \Lambda$ either $c_\alpha p_\alpha \preccurlyeq c_\beta p_\beta$ or $c_\beta p_\beta \preccurlyeq c_\alpha p_\alpha$. So is the family $\{np_n\colon n \in \mathbb{N}\}$ if V has a countable saturated family $\{p_n\}$ of matrix seminorms, that is, V is a Fréchet operator space. We re-denote $c_\alpha p_\alpha$ again by p_α, and put α instead of (α, c_α), respectively. By Proposition 6.1, we derive the inclusions $\mathfrak{s}_\alpha \subseteq \mathfrak{s}_\beta$ or $\mathfrak{s}_\beta \subseteq \mathfrak{s}_\alpha$, which in turn implies that $P_\alpha \leqslant P_\beta$ or $P_\beta \leqslant P_\alpha$ (see the proof of Theorem 7.1). Then

$$\Phi(V) \subseteq C^*_{\mathcal{E}}(\mathcal{D}).$$

Indeed, one should prove that $P_\alpha \Phi(v) \subseteq \Phi(v) P_\alpha$ for all $v \in V$ and $\alpha \in \Lambda$. Take $x \in \mathcal{D}$. Then $x = (x_w)_{w \in \mathfrak{s}_\beta} \in H_\beta$ $(x_w \in \mathbb{C}^{n(w)})$ for some $\beta \in \Lambda$. If $\alpha \preccurlyeq \beta$ then $P_\alpha \Phi(v)x = P_\alpha(w(v)x_w)_{w\in\mathfrak{s}_\beta} = (w(v)x_w)_{w\in\mathfrak{s}_\alpha} = \Phi(v)P_\alpha x$. But if $\alpha \succeq \beta$ then $x = P_\alpha x$ and $P_\alpha \Phi(v)x = \Phi(v)x = \Phi(v)P_\alpha x$. Thus $P_\alpha \Phi(v) \subseteq \Phi(v) P_\alpha$ for all α, that is, $\Phi(V) \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ (see (3.2)).

If A is a multinormed C^*-algebra then A is a local operator space (see Section 2.2). Using the Representation theorem 7.1, we conclude that there is a topological matrix embedding $\varphi: A \to C_{\mathcal{E}}(\mathcal{D})$. If $\varphi: A \to C_{\mathcal{E}}(\mathcal{D})$ preserves C^*-operations, that is, $\varphi(A) \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ and φ is a unital $*$-isomorphism, then A is a local operator C^*-algebra on H, and the local positivity in A would be reduced to the same one but in the local operator system sense. That is indeed true as follows from the next representation theorem.

7.2. *The representation theorem for multinormed C^*-algebras*

Now we propose a representation theorem for multinormed C^*-algebras.

Theorem 7.2. *Let A be a unital multinormed C^*-algebra with a defining family $\{p_\alpha\colon \alpha \in \Lambda\}$ of C^*-seminorms. There is a local isometrical $*$-homomorphism $A \to C^*_{\mathcal{E}}(\mathcal{D})$ for some quantized domain $\mathcal{E}$ with its union space $\mathcal{D}$. In particular, A is a local operator C^*-algebra on a certain quantized domain up to a topological* (*matrix*) *$*$-isomorphism.*

Proof. Take $\alpha \in \Lambda$. Let $H_\alpha = \bigoplus_{\pi \in \mathfrak{s}_\alpha} H_\pi$ be the Hilbert space direct sum over the set $\mathfrak{s}_\alpha$ from Section 6.2, where H_π is the representation space of π, $\alpha \in \Lambda$. Note that H_α is a closed subspace in H_β whenever $p_\alpha \leqslant p_\beta$, $\alpha, \beta \in \Lambda$, for $\mathfrak{s}_\alpha \subseteq \mathfrak{s}_\beta$. Moreover, all H_α are closed subspaces of the Hilbert space $H = \bigoplus_{\pi \in \mathfrak{s}} H_\pi$. Consider the quantized domain $\mathcal{E} = \{H_\alpha\}_{\alpha\in\Lambda}$ in H and put $\mathcal{D} = \bigcup \mathcal{E}$. Let us introduce a linear mapping

$$\Phi_\alpha : A \to \mathcal{B}(H_\alpha), \quad \Phi_\alpha(a) = \bigl(\pi(a)\bigr)_{\pi \in \mathfrak{s}_\alpha}.$$

Using Proposition 6.2, we deduce that

$$\bigl\|\Phi_\alpha(a)\bigr\|_{\mathcal{B}(H_\alpha)} = \sup\bigl\{\bigl\|\pi(a)\bigr\|_{\mathcal{B}(H_\pi)}\colon \pi \in \mathfrak{s}_\alpha\bigr\} = p_\alpha(a)$$

for all $a \in A$. Evidently, Φ_α is a $*$-representation. Moreover, $\Phi_\beta(a)|H_\alpha = \Phi_\alpha(a)$ if $\alpha \leqslant \beta$. So, we have a well-defined unbounded operator $\Phi(a)$ on H with domain $\mathcal{D}$ such that $\Phi(a)(H_\alpha) \subseteq H_\alpha$, $\Phi(a)|H_\alpha = \Phi_\alpha(a)$, $\alpha \in \Lambda$, that is, $\Phi(a) \in C_{\mathcal{E}}(\mathcal{D})$. Note that if $\Phi(a) = 0$ then $\Phi_\alpha(a) = 0$ for all α. Hence $p_\alpha(a) = 0$ for all α, and therefore $a = 0$. Hence we have a linear embedding

$$\Phi : A \to C_{\mathcal{E}}(\mathcal{D}), \quad a \mapsto \Phi(a).$$

Take $a \in A$. Since each Φ_α is a $*$-representation, it follows that $\Phi(a)$ admits an unbounded dual $\Phi(a)^\bigstar$ such that $\mathcal{D} \subseteq \mathrm{dom}(\Phi(a)^\bigstar)$, $\Phi(a)^\bigstar(\mathcal{D}) \subseteq \mathcal{D}$ and $\Phi(a)^* = \Phi(a)^\bigstar|\mathcal{D} = \Phi(a^*) \in C_\mathcal{E}(\mathcal{D})$. By Proposition 3.1, $\Phi(a) \in C^*_\mathcal{E}(\mathcal{D})$. Thus $\Phi : A \to C^*_\mathcal{E}(\mathcal{D})$ is an injective $*$-homomorphism. Moreover, Φ is a local isometric mapping, for $p_\alpha(a) = \|\Phi_\alpha(a)\|_{\mathcal{B}(H_\alpha)} = \|\Phi(a)|H_\alpha\|_{\mathcal{B}(H_\alpha)}$.

Finally, note that Φ is a local matrix isometry. Indeed, there is a well-defined unital $*$-homomorphism $\phi_\alpha : A_\alpha \to \mathcal{B}(H_\alpha)$ such that $\phi_\alpha(\pi_\alpha(a)) = \Phi(a)|H_\alpha$. It follows that ϕ_α is a matrix isometry. Moreover, $\phi_\alpha^{(n)}(\pi_\alpha^{(n)}(a)) = \Phi^{(n)}(a)|H_\alpha^n$, $a \in M_n(A)$. Thus

$$\left\|\Phi^{(n)}(a)|H_\alpha^n\right\| = \left\|\phi_\alpha^{(n)}\left(\pi_\alpha^{(n)}(a)\right)\right\| = \left\|\pi_\alpha^{(n)}(a)\right\|_{M_n(A_\alpha)} = p_\alpha^{(n)}(a)$$

for all $a \in M_n(A)$. Whence Φ is a local matrix isometry. □

Corollary 7.1. *Let A and B be unital multinormed C^*-algebras and let $\varphi : A \to B$ be a unital linear isomorphism between them. If both φ and φ^{-1} are local matrix positive, then φ is a topological $*$-isomorphism. In particular, if φ is a local matrix isometry then φ is a local isometrical $*$-isomorphism.*

Proof. By Theorem 7.2, one may assume that $A \subseteq C^*_\mathcal{S}(\mathcal{O})$ and $B \subseteq C^*_\mathcal{E}(\mathcal{D})$ are local operator C^*-algebras on some quantized domains $\mathcal{S}$ and $\mathcal{E}$, respectively. Let $\mathfrak{q} = \{Q_\alpha\}_{\alpha\in\Omega}$ and $\mathfrak{p} = \{P_\iota\}_{\iota\in\Lambda}$ be the projection nets associated with $\mathcal{S}$ and $\mathcal{D}$, respectively. Using Corollaries 4.1 and 5.5, infer that $\varphi(a)^*\varphi(a) \leqslant \varphi(a^*a)$, $a \in A$ (respectively, $\varphi^{-1}(b)^*\varphi^{-1}(b) \leqslant \varphi^{-1}(b^*b)$, $b \in B$) on $\mathcal{D}$ (respectively, on $\mathcal{O}$). Take an index $\alpha \in \Omega$. Since φ^{-1} is local positive, it follows that there is $\iota \in \Lambda$ such that $\varphi^{-1}(b) \geqslant_\alpha 0$ if $b \geqslant_\iota 0$. In particular, $\varphi(a)^*\varphi(a) \leqslant_\iota \varphi(a^*a)$ implies that $\varphi^{-1}(\varphi(a)^*\varphi(a)) \leqslant_\alpha \varphi^{-1}(\varphi(a^*a))$, or $\varphi^{-1}(\varphi(a)^*\varphi(a))Q_\alpha \leqslant a^*aQ_\alpha$. But $\varphi^{-1}(\varphi(a)^*)\varphi^{-1}(\varphi(a)) \leqslant \varphi^{-1}(\varphi(a)^*\varphi(a))$ on $\mathcal{D}$, in particular, $a^*aQ_\alpha \leqslant \varphi^{-1}(\varphi(a)^*\varphi(a))Q_\alpha$. Being a^*aQ_α and $\varphi^{-1}(\varphi(a)^*\varphi(a))Q_\alpha$ hermitian elements, we conclude that $a^*aQ_\alpha = \varphi^{-1}(\varphi(a)^*\varphi(a))Q_\alpha$. Hence $\varphi(a^*a) = \varphi(a)^*\varphi(a)$ for all $a \in A$. Using Corollary 5.5 again, we obtain that φ is an algebra homomorphism.

Finally, note that if φ is a unital local matrix isometry then automatically φ and φ^{-1} are local matrix positive by virtue of Corollary 4.1. □

8. Injectivity

In this section we prove Hahn–Banach extension theorem for local operator spaces and local operator systems. As a corollary we obtain that the Arens–Michael algebra $C_\mathcal{E}(\mathcal{D})$ is an injective local operator space whereas $C^*_\mathcal{E}(\mathcal{D})$ is an injective object in both local operator space and local operator system senses.

8.1. Injective and strong injective local operator spaces

A (Hausdorff) local operator space V is said to be *an injective local operator space* if for any subspace $W_0 \subseteq W$ of a local operator space W, every matrix continuous mapping $\varphi : W_0 \to V$ can be extended up to a matrix continuous mapping $\psi : W \to V$. The following strong version of the injectivity plays an important role. Let V be a local operator space with a defining family of matrix seminorms $\{p_\alpha\colon \alpha \in \Lambda\}$. Consider a local operator space W, its subspace W_0 and a local matrix contractive mapping $\varphi : W_0 \to V$ with respect to the family $\{p_\alpha\colon \alpha \in \Lambda\}$ on V and a certain family $\{q_\iota\colon \iota \in \Omega\}$ of matrix seminorms on W. Hence for each $\alpha \in \Lambda$ relates $\iota \in \Omega$ with $p_\alpha^{(\infty)}\varphi^{(\infty)} \leqslant q_\iota^{(\infty)}|_{\mathbb{M}(W_0)}$. If φ is extended up to a local matrix contractive mapping $\psi: W \to V$ with respect to the same families $\{p_\alpha\colon \alpha \in \Lambda\}$ and $\{q_\iota\colon \iota \in \Omega\}$, then V is called a *strong injective local operator space*. Consequently, for each $\alpha \in \Lambda$ relates $\kappa \in \Omega$ (not necessarily the same ι) with $p_\alpha^{(\infty)}\psi^{(\infty)} \leqslant q_\kappa^{(\infty)}$.

Obviously, strong injectivity implies injectivity. Moreover, $V = \mathcal{B}(H)$ is a strong injective local operator space with respect to the operator norm. Indeed, if $\varphi: W_0 \to \mathcal{B}(H)$ is a matrix contractive mapping with $\|\varphi^{(\infty)}(a)\| \leqslant q_\iota^{(\infty)}(a)$, $a \in M(W_0)$, for some continuous matrix seminorm q_ι on the local operator space W, then using Hahn–Banach theorem [20, Theorem 2.3.1], we have an extension $\psi: W \to \mathcal{B}(H)$ of φ such that

$$\left\|\psi^{(\infty)}(a)\right\| \leqslant q_\iota^{(\infty)}(a), \quad a \in M(W).$$

The latter in turn implies that ψ is a local matrix contractive mapping, that is, $\mathcal{B}(H)$ is a strong injective local operator space.

Lemma 8.1. *Let V be an injective local operator space and let $\Phi: V \to V$ be a matrix continuous projection. Then $\Phi(V)$ is an injective local operator space. Moreover, if V is strong injective with respect to a defining family $\{p_\alpha\colon \alpha \in \Lambda\}$ of matrix seminorms and $\Phi: V \to V$ is a local matrix contraction with respect to $\{p_\alpha\colon \alpha \in \Lambda\}$, then $\Phi(V)$ is strong injective with respect to $\{p_\alpha\colon \alpha \in \Lambda\}$.*

The proof directly follows from the relevant definitions.

Theorem 8.1. *Let $\mathcal{E} = \{H_n\}_{n\in\mathbb{N}}$ be a quantized Fréchet domain in a Hilbert space H with its union space $\mathcal{D}$. Then $C_{\mathcal{E}}(\mathcal{D})$ is an injective local operator space and $C^*_{\mathcal{E}}(\mathcal{D})$ is a strong injective local operator space.*

Proof. We reduce the problem to the case when $\mathcal{E} = \{H\}$, that is, $C_{\mathcal{E}}(\mathcal{D}) = \mathcal{B}(H)$. As we have shown above the assertion is true for that case, confirming that $\mathcal{B}(H)$ is an injective local operator space, we shall end the proof.

Let $W_0 \subseteq W$ be local operator spaces and let $\varphi: W_0 \to C_{\mathcal{E}}(\mathcal{D})$ be a matrix continuous mapping. Let us prove that φ has a matrix continuous extension $\Phi: W \to C_{\mathcal{E}}(\mathcal{D})$. By assumption $C_{\mathcal{E}}(\mathcal{D})$ is the Fréchet–Arens–Michael algebra (see Proposition 4.1). Let $\mathfrak{p} = \{P_n\}_{n\in\mathbb{N}}$ be the projection net in $\mathcal{B}(H)$ associated with $\mathcal{E}$. Fix $n \in \mathbb{N}$ and consider the linear mapping $\varphi_n: W_0 \to \mathcal{B}(H)$, $\varphi_n(w) = P_n\varphi(w)P_n$. Being $\varphi: W_0 \to C_{\mathcal{E}}(\mathcal{D})$ matrix continuous, we deduce that $\|\varphi^{(s)}(w)|H_n^s\| \leqslant q_{\iota_n}^{(s)}(w)$ for all $w \in M_s(W_0)$, $s \in \mathbb{N}$, where $\{q_\iota\}_{\iota\in\Omega}$ is a certain defining (saturated) family of matrix seminorms on W, $\iota_n \in \Omega$, $n \in \mathbb{N}$. Taking into account that $\varphi_n^{(s)}(w) = P_n^{\oplus s}\varphi^{(s)}(w)P_n^{\oplus s}$, we deduce that $\|\varphi_n^{(s)}(w)\| \leqslant q_{\iota_n}^{(s)}(w)$, $w \in M_s(W_0)$, $s \in \mathbb{N}$, that is, each φ_n is matrix continuous. Being $\mathcal{B}(H)$ a strong injective local operator space, we conclude that φ_n has a matrix continuous extension $\Phi_n: W \to \mathcal{B}(K)$ with $\|\Phi_n^{(\infty)}(v)\| \leqslant q_{\iota_n}^{(\infty)}(v)$, $v \in M(W)$. Consider the linear mapping

$$\Phi: W \to C_{\mathcal{E}}(\mathcal{D}), \quad \Phi(v) = \sum_{m=1}^{\infty}\sum_{k=1}^{m} S_k\Phi_m(v)S_m.$$

Note that $\Phi(v)P_n = \sum_{m=1}^{n}\sum_{k=1}^{m} S_k\Phi_m(v)S_m$ (see Proposition 4.2). Thus $\Phi(v)(H_n) \subseteq H_n$ and $\Phi(v)|H_n \in \mathcal{B}(H_n)$ for all n, that is, Φ is well defined. Let us verify that $\Phi: W \to C_{\mathcal{E}}(\mathcal{D})$ is matrix continuous. First note that $\Phi^{(s)}(v)=\sum_{m=1}^{\infty}\sum_{k=1}^{m}S_k^{\oplus s}\Phi_m^{(s)}(v)S_m^{\oplus s}$. Further, fix $n\in\mathbb{N}$. For each m, $m\leqslant n$, there corresponds an index $\iota_m\in\Omega$ such that $\|\Phi_m^{(\infty)}(v)\|\leqslant q_{\iota_m}^{(\infty)}(v)$, $v\in M(W)$. Take $\iota\in\Omega$ with $\iota\geqslant\iota_m$, $m\leqslant n$, and let $v\in M_s(W)$. Then

$$\left\|\Phi^{(s)}(v)|K_n^s\right\|=\left\|\sum_{m=1}^{n}\sum_{k=1}^{m}S_k^{\oplus s}\Phi_m^{(s)}(v)S_m^{\oplus s}\right\|\leqslant\sum_{m=1}^{n}\left\|P_m^{\oplus s}\Phi_m^{(s)}(v)S_m^{\oplus s}\right\|\leqslant nq_\iota^{(s)}(v).$$

Thus $\Phi:W\to C_{\mathcal{E}}(\mathcal{D})$ is matrix continuous.

If $\varphi(W_0)\subseteq C^*_{\mathcal{E}}(\mathcal{D})$ and $\varphi:W_0\to C^*_{\mathcal{E}}(\mathcal{D})$ is local matrix contractive, then we consider the linear mapping $\Psi:W\to C^*_{\mathcal{E}}(\mathcal{D})$, $\Psi=D\cdot\Phi$, where $D:C_{\mathcal{E}}(\mathcal{D})\to C_{\mathcal{E}}(\mathcal{D})$ is the local matrix contractive projection from Proposition 4.2. Then $\Psi(v)=\sum_{m=1}^{\infty}S_m\Phi_m(v)S_m$ and

$$\begin{aligned}\left\|\Psi^{(s)}(v)|H_n^s\right\|&=\left\|\bigoplus_{m=1}^{n}S_m^{\oplus s}\Phi_m^{(s)}(v)S_m^{\oplus s}\right\|=\max\left\{\left\|S_m^{\oplus s}\Phi_m^{(s)}(v)S_m^{\oplus s}\right\|\colon m\leqslant n\right\}\\&\leqslant\max\left\{\left\|\Phi_m^{(s)}(v)\right\|\colon m\leqslant n\right\}\leqslant\max\left\{q_{\iota_m}^{(s)}(v)\colon m\leqslant n\right\}\\&\leqslant q_\iota^{(s)}(v),\end{aligned}$$

that is, Ψ is a local matrix contractive mapping.

It remains to observe that Φ (respectively, Ψ) extends the mapping $\varphi:W_0\to C_{\mathcal{E}}(\mathcal{D})$. Take $w\in W_0$. Using Proposition 4.2, we deduce that

$$\Phi(w)P_n=\sum_{m=1}^{n}\sum_{k=1}^{m}S_k\Phi_m(v)S_m=\sum_{m=1}^{n}\sum_{k=1}^{m}S_kP_m\varphi(w)P_mS_m=\sum_{m=1}^{n}\sum_{k=1}^{m}S_k\varphi(w)S_m=\varphi(w)P_n$$

for all n. Hence $\Phi(w)=\varphi(w)$. Furthermore, $\Psi(w)=D(\Phi(w))=D(\varphi(w))=\varphi(w)$ whenever $\varphi(W_0)\subseteq C^*_{\mathcal{E}}(\mathcal{D})$. Thus $C_{\mathcal{E}}(\mathcal{D})$ (respectively, $C^*_{\mathcal{E}}(\mathcal{D})$) is an injective (respectively, a strong injective) local operator space. □

Corollary 8.1. *Let V be a Fréchet operator space. Then V is* (*strong*) *injective if and only if it is the range of a matrix continuous* (*local matrix contractive*) *projection $C^*_{\mathcal{E}}(\mathcal{D})\to C^*_{\mathcal{E}}(\mathcal{D})$ for a certain Fréchet quantized domain $\mathcal{E}$, up to a topological matrix isomorphism.*

Proof. Using Representation theorem 7.1 and Remark 7.2, infer that $V\subseteq C^*_{\mathcal{E}}(\mathcal{D})$ for a certain Fréchet domain $\mathcal{E}$. If V is (strong) injective then the identity mapping $V\to V$ is extended up to a matrix continuous (local matrix contractive) projection $\Phi:C^*_{\mathcal{E}}(\mathcal{D})\to V$. Conversely, $C^*_{\mathcal{E}}(\mathcal{D})$ is a strong injective local operator space thanks to Theorem 8.1. Using Lemma 8.1, we conclude that V is (strong) injective whenever V is the range of a matrix continuous (local matrix contractive) projection on $C^*_{\mathcal{E}}(\mathcal{D})$. □

8.2. Arveson–Hahn–Banach–Webster theorem for local operator systems

In this subsection we introduce injective local operator systems and prove a locally convex version of Arveson–Hahn–Banach–Wittstock theorem for operator systems.

As in the normed case [9, Chapter 6] we say that V is *an injective local operator system* if every morphism (unital local matrix positive mapping) $\varphi_0 : W_0 \to V$ can be extended up to a morphism $\varphi : W \to V$ for a local operator system W and its operator system subspace W_0.

Theorem 8.2. *Let $\mathcal{E}$ be a quantized Fréchet domain with its union space $\mathcal{D}$ and let $V \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ be a Fréchet operator system. The following assertions are equivalent*:

(i) *V is an injective local operator system*;
(ii) *there is a morphism-projection $C^*_{\mathcal{E}}(\mathcal{D}) \to C^*_{\mathcal{E}}(\mathcal{D})$ onto V*;
(iii) *V is a strong injective local operator space*;
(iv) *if $W_0 \subseteq W$ are local operator systems and $\varphi_0 : W_0 \to V$ is a local matrix contractive mapping, then it has a local matrix contractive extension $\varphi : W \to V$.*

In particular, $C^*_{\mathcal{E}}(\mathcal{D})$ is an injective local operator system.

Proof. If V is an injective local operator system then the identity mapping $V \to V$ is extended up to a morphism-projection $C^*_{\mathcal{E}}(\mathcal{D}) \to V$ by its very definition, that is, we have (i) $\Rightarrow$ (ii).

Now assume that $\Phi : C^*_{\mathcal{E}}(\mathcal{D}) \to C^*_{\mathcal{E}}(\mathcal{D})$ is a morphism-projection onto V. By Corollary 4.1, Φ is a local matrix contraction. From Lemma 8.1 and Theorem 8.1, we derive that V is a strong injective local operator space, that is, (ii) $\Rightarrow$ (iii).

The implication (iii) $\Rightarrow$ (iv) is trivial.

In order to prove the implication (iv) $\Rightarrow$ (i), assume that W is a local operator system with its operator system subspace W_0 and $\varphi_0 : W_0 \to V$ is a morphism. By Corollary 4.1, φ_0 is a local matrix contraction. By assumption it has a local matrix contractive extension $\varphi : W \to V$. Being φ a unital mapping, we deduce that φ is a morphism. Whence V is injective as a local operator system. $\square$

Thus the injectivity for Fréchet operator systems is reduced to the *strong injectivity* in the class of all local operator spaces.

8.3. The $\widehat{\otimes}^*$-algebra structure on an injective local operator system

Let $V \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ be an injective local operator system on a quantized domain $\mathcal{E} = \{H_\alpha\}_{\alpha \in \Lambda}$ with its union space $\mathcal{D}$. There is a morphism-projection $\Phi : C^*_{\mathcal{E}}(\mathcal{D}) \to C^*_{\mathcal{E}}(\mathcal{D})$onto V (see to the proof of Theorem 8.2). In particular, V is a complete local operator system. For $T, S \in V$ we put

$$T \cdot S = \Phi(TS). \tag{8.1}$$

It is a well-defined bilinear mapping $V \times V \to V$. Since $T = \Phi(T)$, $S = \Phi(S)$, it follows that

$$\begin{aligned} T \cdot (S \cdot R) &= \Phi\big(T\Phi(SR)\big) = \Phi\big(\Phi(T)SR\big) = \Phi(TSR) = \Phi\big(TS\Phi(R)\big) = \Phi\big(\Phi(TS)R\big) \\ &= (T \cdot S) \cdot R \end{aligned}$$

by virtue of Corollary 5.6. Furthermore, taking into account that Φ is $*$-linear (see Lemma 4.3), we conclude that $(T \cdot S)^* = \Phi(TS)^* = \Phi((TS)^*) = \Phi(S^*T^*) = S^* \cdot T^*$. Fix $\alpha \in \Lambda$ and $T \in V$. Then $T = \Phi(T)$ and $T^*T = \Phi(T)^*\Phi(T) \leqslant \Phi(T^*T)$ on $\mathcal{D}$, thanks to Corollary 5.5. In particular,

$T^*T \leqslant_\alpha \Phi(T^*T)$. It follows that $p_\alpha^{(1)}(T)^2 = p_\alpha^{(1)}(T^*T) \leqslant p_\alpha^{(1)}(T^* \cdot T)$. Being Φ a local matrix contraction, we conclude that $p_\alpha^{(\infty)}\Phi^{(\infty)} \leqslant p_\beta^{(\infty)}$ for a certain $\beta \in \Lambda$, $\beta \geqslant \alpha$. Thus $p_\alpha^{(1)}(T^* \cdot T) = p_\alpha^{(1)}(\Phi(T^*T)) \leqslant p_\beta^{(1)}(T^*T) = p_\beta^{(1)}(T)^2$, that is,

$$p_\alpha^{(1)}(T)^2 \leqslant p_\alpha^{(1)}(T^* \cdot T) \leqslant p_\beta^{(1)}(T)^2$$

for all $T \in V$. Moreover,

$$p_\alpha^{(1)}(T \cdot S) = p_\alpha^{(1)}\big(\Phi(TS)\big) \leqslant p_\beta^{(1)}(TS) \leqslant p_\beta^{(1)}(T)p_\beta^{(1)}(S),$$

$T, S \in V$, that is, V is a unital $\widehat{\otimes}^*$-algebra (locally convex $*$-algebra with the jointly continuous multiplication).

Now consider the morphism-projection $\Phi^{(n)} : M_n(C_\mathcal{E}^*(\mathcal{D})) \to M_n(C_\mathcal{E}^*(\mathcal{D}))$. Then $\mathrm{im}(\Phi^{(n)}) = M_n(V)$. Moreover, the multiplication on $M_n(V)$ induced by the projection $\Phi^{(n)}$ coincides with the matrix multiplication inherited by one on V. Indeed, take $a = [a_{ij}]$, $b = [b_{ij}] \in M_n(V)$. Then

$$a \cdot b = \left[\sum_k a_{ik} \cdot b_{kj}\right] = \left[\sum_k \Phi(a_{ik}b_{kj})\right] = \Phi^{(n)}(ab).$$

As above, we conclude that

$$p_\alpha^{(n)}(a)^2 \leqslant p_\alpha^{(n)}(a^* \cdot a) \leqslant p_\beta^{(n)}(a)^2 \quad \text{and} \quad p_\alpha^{(n)}(a \cdot b) \leqslant p_\beta^{(n)}(a)p_\beta^{(n)}(b)$$

(with the same α, β) for all $a, b \in M_n(V)$.

Now we investigate when V with its $\widehat{\otimes}^*$-algebra structure turns out to be a multinormed C^*-algebra such that their local operator space structures are equivalent.

Lemma 8.2. *For each $\alpha \in \Lambda$ there corresponds $\beta \in \Lambda$, $\beta \geqslant \alpha$, such that for all $b, c \in M_n(V)$, $c \geqslant_\beta 0$, we have $b^* \cdot c \cdot b \geqslant_\alpha 0$.*

Proof. As we have just indicated above for each $\alpha \in \Lambda$ there corresponds $\beta \in \Lambda$, $\beta \geqslant \alpha$, such that $p_\alpha^{(\infty)}\Phi^{(\infty)} \leqslant p_\beta^{(\infty)}$, that is, the canonical mapping $V_\beta \to V_\alpha$ between the operator systems V_β and V_α (see Remark 4.1) induced by Φ is a matrix contraction. Since $V_\beta \to V_\alpha$ is unital, we conclude that $V_\beta \to V_\alpha$ is matrix positive, that is, if $a \geqslant_\beta 0$ for some $a \in M_n(V)$ then $\Phi^{(n)}(a) \geqslant_\alpha 0$. Take $c \in M_n(V)$ with $c \geqslant_\beta 0$. Then $b^*cb \geqslant_\beta 0$ in $M_n(C_\mathcal{E}^*(\mathcal{D}))$, and therefore $\Phi^{(n)}(b^*cb) \geqslant_\alpha 0$. It remains to note that $b^* \cdot c \cdot b = \Phi^{(n)}(\Phi^{(n)}(b^*c)b) = \Phi^{(n)}(b^*c\Phi^{(n)}(b)) = \Phi^{(n)}(b^*cb)$ thanks to Corollary 5.6. $\square$

As in the proof of Stinespring theorem 5.1, we extend the quantized domain $\mathcal{E}$ to decompose the operators from V. Confirm that V is just a unital $\widehat{\otimes}^*$-algebra. Consider the algebraic tensor product $V \otimes \mathcal{D}$ and let $\langle\cdot,\cdot\rangle$ be the sesquilinear form on $V \otimes \mathcal{D}$ determined by the rule

$$\left\langle \sum b_j \otimes \eta_j, \sum a_i \otimes \xi_i \right\rangle = \sum \langle (a_i^* \cdot b_j)\eta_j | \xi_i \rangle = \sum \langle \Phi(a_i^* b_j)\eta_j | \xi_i \rangle$$

for all $a_i, b_j \in A$ and $\xi_i, \eta_j \in \mathcal{D}$, where $\langle\cdot|\cdot\rangle$ is the original Hilbert space inner product in H. Using Lemma 5.1 applied to the mapping $\Phi : A(V) \to C_\mathcal{E}^*(\mathcal{D})$ form the multinormed C^*-algebra

$A(V)$ in $C^*_{\mathcal{E}}(\mathcal{D})$ generated by V into the algebra $C^*_{\mathcal{E}}(\mathcal{D})$, we conclude that $\langle\cdot,\cdot\rangle$ is a positive semidefined sesquilinear form on $V\otimes\mathcal{D}$. Therefore it induces a Hilbert space inner product on the quotient space $(V\otimes\mathcal{D})/N$ modulo the subspace $N=\{u\in V\otimes\mathcal{D}\colon \langle u,u\rangle=0\}$. In particular, $\langle x,y\rangle=\sum_{j=1}^n\langle u_j,v_j\rangle$, $x=(u_j)$, $y=(v_j)\in(V\otimes\mathcal{D})^n$, is a positive semidefined sesquilinear form on $(V\otimes\mathcal{D})^n$.

Let K be the completion of the pre-Hilbert space $(V\otimes\mathcal{D})/N$, M_α the range of the subspace $V\otimes H_\alpha$ via the quotient mapping $(V\otimes\mathcal{D})\to(V\otimes\mathcal{D})/N$, and let K_α be the closure of M_α in K. Evidently, $\mathcal{S}=\{K_\alpha\}_{\alpha\in\Lambda}$ is a quantized domain in K and let $\mathcal{O}$ be its union space. Furthermore, the mapping $\mathcal{D}\to K$, $\xi\mapsto(1_{\mathcal{D}}\otimes\xi)^\sim$, is an isometry. Indeed,

$$\left\|(1_{\mathcal{D}}\otimes\xi)^\sim\right\|_K^2=\left\langle(1_{\mathcal{D}}\otimes\xi)^\sim,(1_{\mathcal{D}}\otimes\xi)^\sim\right\rangle=\langle 1_{\mathcal{D}}\otimes\xi,1_{\mathcal{D}}\otimes\xi\rangle=\langle\xi|\xi\rangle=\|\xi\|_H^2.$$

Then we have an isometrical embedding $T:H\to K$ such that $T\xi=(1_{\mathcal{D}}\otimes\xi)^\sim$, $\xi\in\mathcal{D}$. Evidently, $T(H_\alpha)\subseteq K_\alpha$, $\alpha\in\Lambda$. Thus K is the extension of the Hilbert space H with $H_\alpha\subseteq K_\alpha$, $\alpha\in\Lambda$, that is, $\mathcal{E}\subseteq\mathcal{S}$.

For each $a\in V$, consider the linear mapping $\pi(a)=L_a\otimes I_{\mathcal{D}}:V\otimes\mathcal{D}\to V\otimes\mathcal{D}$, where $L_a\in L(V)$, $L_ax=a\cdot x$, is the left multiplication (by a) operator on V. Evidently, the correspondence $V\to L(V\otimes\mathcal{D})$, $a\mapsto\pi(a)$, is an algebra homomorphism.

Lemma 8.3. *For each $\alpha\in\Lambda$ there corresponds $\beta\in\Lambda$ with $\beta\geqslant\alpha$ and*

$$\left\langle\pi^{(n)}(a)u,\pi^{(n)}(a)u\right\rangle\leqslant p_\beta^{(n)}(a)^2\langle u,u\rangle$$

for all $a\in M_n(V)$, $u\in(V\otimes H_\alpha)^n$, $n\in\mathbb{N}$.

Proof. Fix $\alpha\in\Lambda$ and take $a\in V$ and $u=\sum_{i=1}^m a_i\otimes\xi_i\in V\otimes H_\alpha$ with $\{\xi_i\}\subseteq H_\alpha$, that is, $\xi=(\xi_i)\in H_\alpha^m$. Then

$$\begin{aligned}\left\langle\pi(a)u,\pi(a)u\right\rangle&=\left\langle\sum a\cdot a_i\otimes\xi_i,\sum a\cdot a_i\otimes\xi_i\right\rangle=\left\langle\left[a_i^*\cdot a^*\cdot a\cdot a_j\right]\xi|\xi\right\rangle\\&=\left\langle\left(A^*\cdot\Delta^*\cdot\Delta\cdot A\right)\xi|\xi\right\rangle,\end{aligned}$$

where $\Delta\in M_m(V)$ is the diagonal matrix with the same diagonal element a and

$$A=\begin{bmatrix}a_1 & a_2 & \cdots\\ 0 & 0 & \cdots\\ \vdots & \vdots & \ddots\end{bmatrix}\in M_m(V).$$

Note that $\Delta^*\cdot\Delta=\Phi^{(m)}(\Delta^*\Delta)$ and $\Delta^*\Delta\leqslant_\delta p_\delta^{(1)}(a)^2I_{\mathcal{D}^m}$ for all $\delta\in\Lambda$. By Lemma 8.2, there is an index $\gamma\in\Lambda$, $\gamma\geqslant\alpha$, such that $b^*\cdot c\cdot b\geqslant_\alpha 0$ for all $b,c\in M_m(V)$ with $c\geqslant_\gamma 0$. Since Φ is local matrix positive, it follows that for that index γ there corresponds an index $\beta\in\Lambda$, $\beta\geqslant\gamma$ such that $\Phi^{(m)}(b)\geqslant_\gamma 0$ whenever $b\geqslant_\beta 0$, $b\in M_m(C^*_{\mathcal{E}}(\mathcal{D}))$, $m\in\mathbb{N}$. But $\Delta^*\Delta\leqslant_\beta p_\beta^{(1)}(a)^2I_{\mathcal{D}^m}$, thereby $\Delta^*\cdot\Delta\leqslant_\gamma p_\beta^{(1)}(a)^2I_{\mathcal{D}^m}$. Furthermore, $A^*\cdot\Delta^*\cdot\Delta\cdot A\leqslant_\alpha p_\beta^{(1)}(a)^2A^*\cdot A$. It follows that $\langle\pi(a)u,\pi(a)u\rangle=\langle(A^*\cdot\Delta^*\cdot\Delta\cdot A)\xi|\xi\rangle\leqslant p_\beta^{(1)}(a)^2\langle(A^*\cdot A)\xi|\xi\rangle=p_\beta^{(1)}(a)^2\langle u,u\rangle$, that is,

$\langle \pi(a)u, \pi(a)u\rangle \leqslant p_\beta^{(1)}(a)^2\langle u, u\rangle$. Now let us prove the same inequality for the matrices keeping in mind the same indices α, γ, β, $\alpha \leqslant \gamma \leqslant \beta$. Let $a = [a_{ij}] \in M_n(V)$ and let $u = (u_j) \in (V \otimes H_\alpha)^n$. Assume that $u_j = \sum_{i=1}^m a_i^{(j)} \otimes \xi_i^{(j)}$, where $\xi_j = (\xi_i^{(j)}) \in H_\alpha^m$ for all j. First, let us prove the equality

$$\bigl\langle \pi^{(n)}(a)u, \pi^{(n)}(a)u\bigr\rangle = \bigl\langle \bigl(A^* \cdot (a^* \cdot a \otimes I_m) \cdot A\bigr)\xi \,|\, \xi\bigr\rangle, \tag{8.2}$$

where $\xi = (\xi_j)_{j=1}^n \in H_\alpha^{mn}$,

$$A_j = \begin{bmatrix} a_1^{(j)} & a_2^{(j)} & \cdots & a_m^{(j)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in M_m(V), \qquad A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix} \in M_{mn}(V),$$

and $a^* \cdot a \otimes I_m \in M_{mn}(V)$ ($I_m \in M_m$ is the identity matrix). Note that $a^* \cdot a = [c_{kj}]$ with $c_{kj} = \sum_{i=1}^n a_{ik}^* \cdot a_{ij}$, and $a^* \cdot a \otimes I_m = [c_{kj} I_m]$. Therefore

$$A^* \cdot (a^* \cdot a \otimes I_m) \cdot A = \begin{bmatrix} A_1^* \cdot c_{11} I_m \cdot A_1 & A_1^* \cdot c_{12} I_m \cdot A_2 & \cdots & A_1^* \cdot c_{1n} I_m \cdot A_n \\ A_2^* \cdot c_{21} I_m \cdot A_1 & A_2^* \cdot c_{22} I_m \cdot A_2 & \cdots & A_2^* \cdot c_{2n} I_m \cdot A_n \\ \vdots & \vdots & \ddots & \vdots \\ A_n^* \cdot c_{n1} I_m \cdot A_1 & A_n^* \cdot c_{n2} I_m \cdot A_2 & \cdots & A_n^* \cdot c_{nn} I_m \cdot A_n \end{bmatrix}$$

and $(A^* \cdot (a^* \cdot a \otimes I_m) \cdot A)\xi = (\sum_{j=1}^n (A_k^* \cdot c_{kj} I_m \cdot A_j)\xi_j)_{k=1}^n \in H_\alpha^{mn}$. Furthermore

$$\begin{aligned} A_k^* \cdot c_{kj} I_m \cdot A_j &= \begin{bmatrix} a_1^{(k)*} & 0 & \cdots & 0 \\ a_2^{(k)*} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_m^{(k)*} & 0 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} c_{kj} & 0 & \cdots & 0 \\ 0 & c_{kj} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{kj} \end{bmatrix} \cdot \begin{bmatrix} a_1^{(j)} & a_2^{(j)} & \cdots & a_m^{(j)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} a_1^{(k)*} \cdot c_{kj} \cdot a_1^{(j)} & a_1^{(k)*} \cdot c_{kj} \cdot a_2^{(j)} & \cdots & a_1^{(k)*} \cdot c_{kj} \cdot a_m^{(j)} \\ a_2^{(k)*} \cdot c_{kj} \cdot a_1^{(j)} & a_2^{(k)*} \cdot c_{kj} \cdot a_2^{(j)} & \cdots & a_2^{(k)*} \cdot c_{kj} \cdot a_m^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ a_m^{(k)*} \cdot c_{kj} \cdot a_1^{(j)} & a_m^{(k)*} \cdot c_{kj} \cdot a_2^{(j)} & \cdots & a_m^{(k)*} \cdot c_{kj} \cdot a_m^{(j)} \end{bmatrix}, \end{aligned}$$

which in turn implies that

$$\sum_{j=1}^n \bigl(A_k^* \cdot c_{kj} I_m \cdot A_j\bigr)\xi_j = \left(\sum_{j=1}^n \sum_{s=1}^m a_t^{(k)*} \cdot c_{kj} \cdot a_s^{(j)} \xi_s^{(j)} \right)_{t=1}^m \in H_\alpha^m.$$

If $z = \langle (A^* \cdot (a^* \cdot a \otimes I_m) \cdot A)\xi \,|\, \xi\rangle$ then

$$
\begin{aligned}
z &= \left\langle \left(\sum_{j=1}^{n} \left(A_k^* \cdot c_{kj} I_m \cdot A_j\right)\xi_j \right)_{k=1}^{n} \Big| (\xi_k)_{k=1}^{n} \right\rangle = \sum_{k=1}^{n} \left\langle \sum_{j=1}^{n} \left(A_k^* \cdot c_{kj} I_m \cdot A_j\right)\xi_j | \xi_k \right\rangle \\
&= \sum_{k=1}^{n} \left\langle \left(\sum_{j=1}^{n} \sum_{s=1}^{m} \left(a_t^{(k)*} \cdot c_{kj} \cdot a_s^{(j)}\right)\xi_s^{(j)} \right)_{t=1}^{m} \Big| \left(\xi_t^{(k)}\right)_{t=1}^{m} \right\rangle \\
&= \sum_{k=1}^{n} \sum_{t=1}^{m} \left\langle \sum_{j=1}^{n} \sum_{s=1}^{m} \left(a_t^{(k)*} \cdot c_{kj} \cdot a_s^{(j)}\right)\xi_s^{(j)} | \xi_t^{(k)} \right\rangle = \sum_{k,j=1}^{n} \sum_{s,t=1}^{m} \left\langle \left(a_t^{(k)*} \cdot c_{kj} \cdot a_s^{(j)}\right)\xi_s^{(j)} | \xi_t^{(k)} \right\rangle \\
&= \sum_{i,k,j=1}^{n} \sum_{s,t=1}^{m} \left\langle \left(a_t^{(k)*} \cdot a_{ik}^* \cdot a_{ij} \cdot a_s^{(j)}\right)\xi_s^{(j)} | \xi_t^{(k)} \right\rangle = \sum_{i,k,j=1}^{n} \sum_{s,t=1}^{m} \left\langle \left(\left(a_{ik} \cdot a_t^{(k)}\right)^* \cdot a_{ij} \cdot a_s^{(j)}\right)\xi_s^{(j)} | \xi_t^{(k)} \right\rangle \\
&= \sum_{i,k,j=1}^{n} \sum_{s,t=1}^{m} \left\langle \left(a_{ij} \cdot a_s^{(j)} \otimes \xi_s^{(j)}\right), \left(a_{ik} \cdot a_t^{(k)} \otimes \xi_t^{(k)}\right) \right\rangle \\
&= \sum_{i,k,j=1}^{n} \left\langle \sum_{s=1}^{m} a_{ij} \cdot a_s^{(j)} \otimes \xi_s^{(j)}, \sum_{t=1}^{m} a_{ik} \cdot a_t^{(k)} \otimes \xi_t^{(k)} \right\rangle = \sum_{i,k,j=1}^{n} \left\langle \pi(a_{ij})u_j, \pi(a_{ik})u_k \right\rangle \\
&= \sum_{i=1}^{n} \left\langle \sum_{j=1}^{n} \pi(a_{ij})u_j, \sum_{k=1}^{n} \pi(a_{ik})u_k \right\rangle = \left\langle \left(\sum_{j=1}^{n} \pi(a_{ij})u_j \right)_{i=1}^{n}, \left(\sum_{j=1}^{n} \pi(a_{ij})u_j \right)_{i=1}^{n} \right\rangle \\
&= \left\langle \pi^{(n)}(a)u, \pi^{(n)}(a)u \right\rangle.
\end{aligned}
$$

Thus the equality (8.2) has been proved.

Now let us derive the required inequality. Note that $a^* \cdot a \otimes I_m = \Phi^{(mn)}(a^*a \otimes I_m)$ and $a^*a \otimes I_m \in M_{mn}(C^*_{\mathcal{E}}(\mathcal{D}))$. Using [9, Proposition 2.1.1], we deduce that

$$
\begin{aligned}
p_\delta^{(nm)}(a^*a \otimes I_m) &= \left\| (a^*a \otimes I_m)|H_\delta^{nm} \right\| = \left\| \left((a^*a)|H_\delta^n\right) \otimes I_m \right\| = \left\| (a^*a)|H_\delta^n \right\| = p_\delta^{(n)}(a^*a) \\
&= p_\delta^{(n)}(a)^2
\end{aligned}
$$

for all $\delta \in \Lambda$. Therefore $a^*a \otimes I_m \leqslant_\delta p_\delta^{(n)}(a)^2 I_{\mathcal{D}^{mn}}$ for all $\delta \in \Lambda$. In particular,

$$
a^*a \otimes I_m \leqslant_\beta p_\beta^{(n)}(a)^2 I_{\mathcal{D}^{mn}},
$$

which in turn implies that $a^* \cdot a \otimes I_m = \Phi^{(nm)}(a^*a \otimes I_m) \leqslant_\gamma p_\beta^{(n)}(a)^2 I_{\mathcal{D}^{mn}}$. By Lemma 8.2,

$$
A^* \cdot (a^* \cdot a \otimes I_m) \cdot A \leqslant_\alpha p_\beta^{(n)}(a)^2 A^* \cdot A.
$$

Using (8.2), we derive that

$$
\begin{aligned}
\left\langle \pi^{(n)}(a)u, \pi^{(n)}(a)u \right\rangle &= \left\langle \left(A^* \cdot (a^* \cdot a \otimes I_m) \cdot A\right)\xi | \xi \right\rangle \\
&\leqslant p_\beta^{(n)}(a)^2 \left\langle \left(A^* \cdot A\right)\xi | \xi \right\rangle = p_\beta^{(n)}(a)^2 \langle u, u \rangle,
\end{aligned}
$$

that is, $\langle \pi^{(n)}(a)u, \pi^{(n)}(a)u \rangle \leqslant p_\beta^{(n)}(a)^2 \langle u, u \rangle$. □

Theorem 8.3. *Let V be an injective local operator system in $C^*_{\mathcal{E}}(\mathcal{D})$. Then V possesses unique multinormed C^*-algebra structure with respect to the involution and matrix topology from $C^*_{\mathcal{E}}(\mathcal{D})$.*

Proof. If we have two unital multinormed C^*-algebra structures on V with the same involution and matrix topology from $C^*_{\mathcal{E}}(\mathcal{D})$ then the identity mapping $V \to V$ being a unital local matrix isometry is automatically $*$-isomorphism by virtue of Corollary 7.1, that is, these structures are identical.

As we have observed above the original matrix seminorms $\{p_\alpha\}$ do not satisfy the C^*-seminorm property with respect to the new multiplication (8.1). That is a crucial moment which does not appear in the normed case, for in the latter case $p_\alpha = p_\beta = p$ for all α, β, therefore the same norm p is a C^*-norm and the proof is ended (see [9, 6.1.3]). To overcome this problem in the multinormed case, below we propose a new family of C^*-seminorms, which is equivalent to $\{p_\alpha\}$.

Take $a \in V$. By Lemma 8.3, $\pi(a)(N) \subseteq N$, therefore it determines a linear mapping on the quotient space $(V \otimes \mathcal{D})/N$ denoted by $\pi(a)$ too. Moreover, $\pi(a)$ leaves invariant each subspace M_α, and for each α relates $\beta \geqslant \alpha$ such that $\|\pi(a)u\| \leqslant p^{(1)}_\beta(a)\|u\|$, $u \in M_\alpha$, thanks to Lemma 8.3. Taking into account that $\overline{M_\alpha} = K_\alpha$, we conclude that the latter inequality is true for all $u \in K_\alpha$. Therefore $\pi(a)$ is extended up to a bounded linear operator on K_α, which we denote by $\pi(a)_\alpha$, and $\|\pi(a)_\alpha\|_{\mathcal{B}(K_\alpha)} \leqslant p^{(1)}_\beta(a)$, $a \in V$. If $K_\alpha \subseteq K_\delta$ then $(\pi(a)_\delta)|K_\alpha = \pi(a)_\alpha$, which can be verified on the dense subspace M_α. Thus we have a well-defined unbounded operator on K with domain $\mathcal{O}$, denoted again by $\pi(a)$, such that $\pi(a)|K_\alpha = \pi(a)_\alpha$ for all α, that is, $\pi(a) \in C_{\mathcal{S}}(\mathcal{O})$ (see (3.1)). Moreover, if $u^\sim, v^\sim \in M_\alpha$ with $u = \sum b_j \otimes \eta_j$, $v = \sum a_i \otimes \xi_i$, $\{\eta_j, \xi_i\} \subseteq H_\alpha$, then

$$\begin{aligned}\langle \pi(a)u^\sim, v^\sim\rangle &= \langle \pi(a)u, v\rangle = \sum \langle a_i^* \cdot a \cdot b_j\eta_j|\xi_i\rangle = \sum \langle (a^* \cdot a_i)^* \cdot b_j\eta_j|\xi_i\rangle = \langle u, \pi(a^*)v\rangle \\ &= \langle u^\sim, \pi(a^*)v^\sim\rangle,\end{aligned}$$

and by continuity, we obtain that $\langle \pi(a)x, y\rangle = \langle x, \pi(a^*)y\rangle$ for all $x, y \in K_\alpha$. Using Proposition 3.1, $\pi(a) \in C^*_{\mathcal{S}}(\mathcal{O})$ and $\pi(a)^* = \pi(a)$. Thus $\pi : V \to C^*_{\mathcal{S}}(\mathcal{O})$, $a \mapsto \pi(a)$, is a unital $*$-homomorphism. Note that

$$\langle \pi(a)(1_{\mathcal{D}} \otimes \eta)^\sim, (1_{\mathcal{D}} \otimes \xi)^\sim\rangle = \langle a \otimes \eta, 1_{\mathcal{D}} \otimes \xi\rangle = \langle \Phi(a)\eta|\xi\rangle = \langle a\eta|\xi\rangle$$

for all $\xi, \eta \in \mathcal{D}$, which means (as in the Stinespring theorem) that

$$a \subseteq P_H\pi(a) \tag{8.3}$$

up to an isometry (see also Corollary 5.1), where $P_H \in \mathcal{B}(K)$ is the projection onto H, that is, $P_H = TT^*$.

Now we introduce a new family $q_\alpha = (q^{(n)}_\alpha)_{n\in\mathbb{N}}$, $q^{(n)}_\alpha(a) = \|\pi^{(n)}(a)|K^n_\alpha\|_{\mathcal{B}(K^n_\alpha)}$, $a \in M_n(V)$, $\alpha \in \Lambda$, of matrix seminorms on V. Confirm that $q^{(1)}_\alpha$ is a multiplicative C^*-seminorm on V. Indeed, $q^{(1)}_\alpha(a^*) = \|\pi(a^*)|K_\alpha\| = \|\pi(a)^*|K_\alpha\| = q^{(1)}_\alpha(a)$,

$$q_\alpha^{(1)}(a^* \cdot a) = \|\pi(a^* \cdot a)|K_\alpha\| = \|\pi(a)^*\pi(a)|K_\alpha\| = \|(\pi(a)|K_\alpha)^*(\pi(a)|K_\alpha)\|$$
$$= \|\pi(a)|K_\alpha\|^2 = q_\alpha^{(1)}(a)^2,$$

and $q_\alpha^{(1)}(a \cdot b) = \|\pi(a \cdot b)|K_\alpha\| = \|(\pi(a)|K_\alpha)(\pi(b)|K_\alpha)\| \leqslant q_\alpha^{(1)}(a)q_\alpha^{(1)}(b)$ for all $a, b \in V$.

It remains to prove that $\{p_\alpha\}$ and $\{q_\alpha\}$ are equivalent family of matrix seminorms on V. Using Lemma 8.3, infer that for each α there corresponds β, $\beta \geqslant \alpha$, such that $q_\alpha^{(n)}(a) = \|\pi^{(n)}(a)|K_\alpha^n\| \leqslant p_\beta^{(n)}(a)$ for all $a \in M_n(V)$, that is, $q_\alpha \preccurlyeq p_\beta$. Moreover, using (8.3), we conclude that

$$p_\alpha^{(n)}(a) = \|a|H_\alpha^n\|_{\mathcal{B}(H_\alpha^n)} \leqslant \|P_H^{\oplus n}\pi^{(n)}(a)|K_\alpha^n\| \leqslant \|\pi^{(n)}(a)|K_\alpha^n\| = q_\alpha^{(n)}(a)$$

for all $a = [a_{ij}] \in M_n(V)$, $n \in \mathbb{N}$. So, $p_\alpha \preccurlyeq q_\alpha$ for all α, that is, $\{p_\alpha\}$ and $\{q_\alpha\}$ are equivalent.

Thus V turns out to be a unital multinormed C^*-algebra preserving its local operator system structure. □

Acknowledgments

I wish to thank Ş. Alpay, E. Emelyanov, M. Fragoulopoulou, A. Gheondea, A. Pirkovskii, F.-H. Vasilescu and C. Webster for useful discussions the details of the present work. The author also thanks the referee(s) for the proposed comments.

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