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TAYLOR FUNCTIONAL CALCULUS FOR SUPERNILPOTENT LIE ALGEBRA OF OPERATORS

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Dedicated to the memory of Nicolas Bourbaki

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ABSTRACT. The present work is motivated by J.L. Taylor's program on noncommutative holomorphic functional calculus within the Lie algebra framework. We propose a sheaf $\mathfrak{T}_{\mathfrak{g}}$ of germs of formally-radical functions in elements of a finite dimensional nilpotent Lie algebra $\mathfrak{g}$ and prove the functional calculus theorem for an operator family generating a supernilpotent Lie subalgebra based upon the sheaf $\mathfrak{T}_{\mathfrak{g}}$. This calculus extends Taylor's holomorphic functional calculus for a mutually commuting operator family.

KEYWORDS: *Noncommutative holomorphic functions in elements of a Lie algebra, formally-radical functions, noncommutative parametrized complexes, Taylor spectrum, transversality.*

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INTRODUCTION

At the beginning of 70's of the previous century the multi-operator holomorphic functional calculus problem has been solved by J.L. Taylor in [26] based upon the methods of topological homology. The advantageous of the homological approach is to have a general vision to the functional calculus problem for (not necessarily commutative) operator families. So, the basic ingredients of the functional calculus are the following:

(1) a base algebra of "polynomials" $\mathcal{B}$;

(2) a Banach left $\mathcal{B}$-module X (or $\mathcal{B}$-functional calculus on X);

(3) a family of topological algebras $\mathcal{A}(U)$ of "noncommutative functions on domains U" with their canonical homomorphisms $\iota_U : \mathcal{B} \to \mathcal{A}(U)$.

The problem is to select the algebras $\mathcal{A}(U)$ such that the space X turns out to be a left Banach $\mathcal{A}(U)$-module whose new $\mathcal{B}$-module structure via pullback along the homomorphism ι_U is reduced to the original one. Roughly speaking, $\mathcal{B}$-functional calculus on X is extended to $\mathcal{A}(U)$-calculi on X. The family $\mathcal{A}(U)$

of topological algebras is usually motivated by the presence of a certain (Fréchet) sheaf whose sections over all open sets U are the algebras $\mathcal{A}(U)$, respectively. For instance, if $T = (T_1, \dots, T_m)$ is an m-tuple of mutually commuting operators on a Banach space X then the holomorphic functional calculus problem for T can be handled within this framework in the following way: the base algebra $\mathcal{B} = P_m$ is the algebra of all complex polynomials in m-commuting complex variables $z = (z_1, \dots, z_m)$ with its natural $\mathcal{B}$-module structure on X:

$$P_m \times X \to X, \quad (p(z), x) \mapsto p(T)x,$$

and $\mathcal{A}(U)$ are the algebras $\mathcal{O}(U)$ of holomorphic functions on domains $U \subseteq \mathbb{C}^m$, respectively. So, we deal with the sheaf $\mathcal{O}$ of germs of holomorphic functions on $\mathbb{C}^m$. The relevant functional calculus problem was solved by J. Taylor [26] in terms of the joint Taylor spectrum $\sigma(T)$ [24]. Namely, if U is an open subset in $\mathbb{C}^m$ enclosing the Taylor spectrum $\sigma(T)$ then X turns into a left Banach $\mathcal{O}(U)$-module extending the action of P_m on X given by the operator tuple T. For a noncommutative base algebra $\mathcal{B}$ we encounter the following principle problems, the first one is to construct of a (Fréchet) sheaf of "germs of functions in noncommuting variables" which would play the same relation to $\mathcal{B}$ as the sheaf $\mathcal{O}$ did play with respect to P_m, the second one is the noncommutative joint spectrum.

One of the central parts of Taylor's program [26] (see also [16]) in the general framework of "noncommutative holomorphic functional calculus" is to replace $\mathcal{B}$ by the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$. Thanks to the recent achievements [1], [3], [17], [9], [13], [14], [15] in the spectral theory for a nilpotent Lie algebra of operators, the indicated part of Taylor's program for a nilpotent Lie algebra $\mathfrak{g}$ appears to be quite attractive. In particular, we have a well defined Taylor spectrum $\sigma(T)$ of the operator tuple $T = (T_1, \dots, T_m)$ generating a finite dimensional nilpotent Lie algebra, which possesses the spectral mapping properties with respect to the noncommutative polynomials. The construction of the relevant Fréchet sheaf of germs of holomorphic functions in elements of $\mathfrak{g}$ appeared to be more complicated. Some developments toward this problem have been done in [10], [11], [12], [23], [13]. It was proposed a suitable (noncommutative) Fréchet algebra presheaf $\mathcal{O}_{\mathfrak{g}}$ over the character space $\Delta(\mathfrak{g})(= \mathbb{C}^m, m = \dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]))$ of the nilpotent Lie algebra $\mathfrak{g}$, with its all desirable properties. But whether $\mathcal{O}_{\mathfrak{g}}$ is a sheaf remains inexplicit (see [6], [7], [8]). Meanwhile, one may observe that the formal Fréchet completion $\mathfrak{T}_{\mathfrak{g}}$ of $\mathcal{O}_{\mathfrak{g}}$ is a Fréchet algebra sheaf. We are saying that $\mathfrak{T}_{\mathfrak{g}}$ is *a sheaf of germs of formally radical functions in elements of* $\mathfrak{g}$. As a sheaf of the Fréchet spaces, $\mathfrak{T}_{\mathfrak{g}}$ has a relatively simple structure, namely it is just the projective tensor product

$$\mathfrak{T}_{\mathfrak{g}} = \mathcal{O} \widehat{\otimes} \mathbb{C}[[\omega_1, \dots, \omega_k]]$$

of the sheaf $\mathcal{O}$ of germs of usual holomorphic functions and the constant sheaf $\mathbb{C}[[\omega_1, \dots, \omega_k]]$ of all formal power series in k-variables over $\mathbb{C}^m$, where $k = \dim([\mathfrak{g}, \mathfrak{g}])$. The algebraic structure on $\mathfrak{T}_{\mathfrak{g}}(D)$ for a polydisk D is uniquely lifted from the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ which is its proper dense subalgebra (see Subsection 5.1). Note that a similar construction is used in the noncommutative algebraic geometry in [22]. In particular, $\mathfrak{T}_{\mathfrak{g}} = \mathcal{O}$ whenever $\mathfrak{g}$ is a commutative Lie algebra. Such formal completion of $\mathcal{O}_{\mathfrak{g}}$ restricts in turn the relevant class of operator tuples to be considered in the functional calculus problem. That is the class of operator tuples on a Banach spaces X generating *supernilpotent Lie subalgebras* in the Banach algebra $\mathcal{L}(X)$ of all bounded linear operators acting on a complex Banach space X. Namely, let $T = (T_1, \ldots, T_m)$ be a family of operators in $\mathcal{L}(X)$ generating a finite-dimensional nilpotent Lie subalgebra $\mathfrak{g}_T \subseteq \mathcal{L}(X)$. It is known [27] that the Lie ideal $[\mathfrak{g}_T, \mathfrak{g}_T]$ of commutators consists of quasinilpotent operators. We say that T generates *a supernilpotent Lie algebra* $\mathfrak{g}_T$ if $[\mathfrak{g}_T, \mathfrak{g}_T]$ consists of nilpotent operators. Obviously, each mutually commuting operator tuple automatically generates a supernilpotent Lie algebra. Moreover, if X is a finite-dimensional space then each nilpotent Lie subalgebra in $\mathcal{L}(X)$ is a supernilpotent one. The class of noncommutative supernilpotent Lie algebras of operators on an infinite-dimensional Banach space X is sufficiently wider than the class of commutative Lie algebras. If T is an m-tuple of operators on X generating a supernilpotent Lie algebra then X turns out to be a left Banach module over the Fréchet algebra $\mathfrak{T}_{\mathfrak{g}}(\mathbb{C}^m)$ of all global sections of the sheaf $\mathfrak{T}_{\mathfrak{g}}$, that is, all entire formally radical functions act on X. Moreover, $\mathfrak{T}_{\mathfrak{g}}(D)$ possesses the Koszul resolution [8], [12], which is a free $\mathfrak{T}_{\mathfrak{g}}(D)$-bimodule resolution.

In the present paper we investigate the functional calculus problem for an operator family generating a supernilpotent Lie algebra based upon the sheaf $\mathfrak{T}_{\mathfrak{g}}$. We propose some modification of Taylor–Helemskii–Putinar framework of the functional calculus [26], [16], Chapter 6 of [20] (see also Section 2). Within this framework the functional calculus problem for a $\mathfrak{g}$-module X can be formulated in the following way. Let $\mathcal{S}$ be a Fréchet (noncommutative) algebra sheaf over $\mathbb{C}^m$ such that there exists an algebra homomorphism $\iota_U : \mathcal{U}(\mathfrak{g}) \to \mathcal{S}(U)$ for each open subset $U \subseteq \mathbb{C}^m$ compatible with the restriction mappings, and assume that X is equipped with a left Banach module structure over the algebra $\mathcal{S}(\mathbb{C}^m)$ of all global sections of the sheaf $\mathcal{S}$ such that its $\mathfrak{g}$-module structure via pullback along $\iota_{\mathbb{C}^m}$ is reduced to the original one. For which open subsets $U \subseteq \mathbb{C}^m$ the space X turns into a left Banach $\mathcal{S}(U)$-module such that its $\mathcal{S}(\mathbb{C}^m)$-module structure via the restriction mapping $\mathcal{S}(\mathbb{C}^m) \to \mathcal{S}(U)$ is reduced to the previous module structure? As the main result we prove that if T is an operator m-tuple in $\mathcal{L}(X)$ generating a supernilpotent Lie subalgebra, and U is an open subset in $\mathbb{C}^m$ such that $\sigma(T) \subseteq U$, then there exists $\mathfrak{T}_{\mathfrak{g}}(U)$-functional calculus on X, that is, X turns into a left Banach $\mathfrak{T}_{\mathfrak{g}}(U)$-module extending the polynomial $\mathcal{U}(\mathfrak{g})$-calculus on X.

1. PRELIMINARIES

All considered linear spaces are complex and algebras are assumed to be unital and associative. The category (usually we refer as a class) of all Fréchet spaces is denoted by **FS**. Given a linear space X, $\bigwedge X = \bigoplus_{k\geqslant 0} \overset{k}{\bigwedge} X$ is the exterior algebra of X. If $\underline{u} = u_1 \wedge \cdots \wedge u_k \in \overset{k}{\bigwedge} X$ is a k-vector then we use the following notation $\underline{u}_i = u_1 \wedge \cdots \wedge \widehat{u_i} \wedge \cdots \wedge u_k$, for $(k-1)$-vector, where $\widehat{u_i}$ means the omission of the variable u_i. If we omit two variables u_i and u_j, $i < j$, from the expression of $\underline{u}$, the obtained vector is denoted by $\underline{u}_{ij}$. The space of all X-valued polynomials in s variables is denoted by $X[\omega_1, \ldots, \omega_s]$, whereas $X[[\omega_1, \ldots, \omega_s]]$ denotes the space of all X-valued formal power series in s variables, so each of its elements f has the unique formal power series expansion $f = \sum_{J\in\mathbb{Z}_+^s} x_J \omega^J$, where $x_J \in X$, $\omega^J = \omega_1^{j_1} \cdots \omega_s^{j_s}$. If X and Y are Fréchet spaces then the space of all continuous linear mappings $X \to Y$ is denoted by $\mathcal{L}(X, Y)$, we also write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$. We use the conventional notation $X \widehat{\otimes} Y$ for the projective tensor product of these spaces. If $\{p_t : t \in \Lambda\}$ is a defining countable seminorm family in X then the space $X[[\omega_1, \ldots, \omega_s]]$ turns out to be a Fréchet space with its defining seminorm family $\{q_{t,K} : (t, K) \in \Lambda \times \mathbb{Z}_+^s\}$, where $q_{t,K}(f) = \max\{p_t(x_J) : J \leqslant K\}$. One may easily verify that the topology generated by the latter seminorm family is merely the direct product topology of $X^{\mathbb{Z}_+^s}$. In particular, $X[[\omega_1, \ldots, \omega_s]] = X \widehat{\otimes} \mathbb{C}[[\omega_1, \ldots, \omega_s]]$, and if X is a nuclear space then so is the space $X[[\omega_1, \ldots, \omega_s]]$.

Take $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, $r = (r_1, \ldots, r_n) \in \overline{\mathbb{R}}_+^n$, and let $D_{a,r}$ be a polydisk in $\mathbb{C}^n$ of multiradius r centered at a. If $a = 0$ then we write D_r instead of $D_{0,r}$. If X is a Banach space then the space of all X-valued holomorphic functions on an open set U is denoted by $\mathcal{O}(U, X)$. For $X = \mathbb{C}$, we write $\mathcal{O}(U)$ instead of $\mathcal{O}(U, \mathbb{C})$. Remember that $\mathcal{O}(U, X)$ is a Fréchet space and $\mathcal{O}(U)$ is a Fréchet algebra with respect to the compact-open topology, and $\mathcal{O}(U, X) = \mathcal{O}(U) \widehat{\otimes} X$ [18] (see also Chapter 2 of [20]). If $U = D_{a,r}$ and $t \in \mathbb{R}_+^n$ with $0 < t < r$, then the seminorm set

$$\Big\| \sum_J x_J (z-a)^J \Big\|_t = \sum_J \|x_J\|_X t^J, \quad t \in \Lambda, \tag{1.1}$$

on $\mathcal{O}(D_{a,r}, X)$ are equivalent to one associated by the compact-open topology due to the known Cauchy inequality.

The Jacobson radical of an algebra A is denoted by Rad A. The left (respectively, right) multiplication operator on A is denoted by L_a (respectively, R_a), that is, $L_a(x) = ax$ and $R_a(x) = xa$ for all $a, x \in A$. The unit of A is denoted by 1_A. Let A be a Fréchet algebra. A Fréchet space X is said to be a *left Fréchet A-module* if X has a structure of a left A-module such that the mapping $A \times X \to X$, $(a, x) \mapsto a \cdot x$ is jointly (or separately) continuous. By analogy, it is defined a *right* *(bi)module over A*. The category of all left Fréchet A-modules is denoted by A-mod. In the same manner mod-A (respectively, A-mod-A) denotes the category of all right (respectively, bi)modules. The category of all chain complexes in **FS** (respectively, A-mod, mod-A, A-mod-A) is denoted by $\underline{\mathbf{FS}}$ (respectively, $\underline{A\text{-}\mathrm{mod}}$, $\underline{\mathrm{mod}\text{-}A}$, $\underline{A\text{-}\mathrm{mod}\text{-}A}$). For the cochain complexes we use the notations $\overline{\mathbf{FS}}$, $\overline{\mathrm{mod}\text{-}A}$, $\overline{A\text{-}\mathrm{mod}}$ and $\overline{A\text{-}\mathrm{mod}\text{-}A}$, respectively.

1.1. RESOLUTIONS AND TRANSVERSALITY. Let A be a Fréchet algebra. The projective tensor product (over A) of Fréchet modules $X \in$ mod-A and $Y \in A$-mod is denoted by $X\underset{A}{\widehat{\otimes}}Y$. By definition, $X\underset{A}{\widehat{\otimes}}Y$ is the quotient space of $X\widehat{\otimes}Y$ with respect to the closed subspace generated by the elements $x \cdot a \otimes y - x \otimes a \cdot y$, $x \in X$, $y \in Y$, $a \in A$. A module $X \in A$-mod is said to be a *free A-module* if $X = A\widehat{\otimes}E$ for a certain Fréchet space E. The left module structure on $A\widehat{\otimes}E$ is given by the rule: $a \cdot (b \otimes e) = ab \otimes e$, $a, b \in A$, $e \in E$. A module $X \in A$-mod is said to be a *projective A-module* if it is a module summand of a certain free A-module. A chain complex

$$(\mathcal{X}, d) : \cdots \longleftarrow X_{n-1} \overset{d_{n-1}}{\longleftarrow} X_n \overset{d_n}{\longleftarrow} X_{n+1} \longleftarrow \cdots$$

in $\underline{A\text{-}\mathrm{mod}}$ is said to be *admissible* if it splits as a complex in $\underline{\mathbf{FS}}$. A *projective resolution* of an A-module X is a complex $(\mathcal{P}, d)$ in $\underline{A\text{-}\mathrm{mod}}$ with $\mathcal{P}_n = \{0\}$ for $n < 0$, together with a morphism $\varepsilon : \mathcal{P}_0 \to X$ such that the augmented complex

$$0 \leftarrow X \overset{\varepsilon}{\longleftarrow} \mathcal{P}_0 \overset{d_0}{\longleftarrow} \mathcal{P}_1 \overset{d_1}{\longleftarrow} \cdots$$

is admissible, and all $\mathcal{P}_n$ are projective modules. If $F : A\text{-mod} \to B\text{-mod}$ is an additive functor then by F_n we denote the n-th projective derived functor of F, where B is a Fréchet algebra. By its very definition, $F_n(X)$ is just the n-th homology of the complex $(F(\mathcal{P}), F(d))$ for a projective resolution $(\mathcal{P}, d)$ of the module X. Taking into account that all projective resolutions of a module are homotopy equivalent (see 3.2.3 of [20]), we conclude that $F_n(X)$ does not depend upon the particular choice of a projective resolution $(\mathcal{P}, d)$ of X. If $F = X\underset{A}{\widehat{\otimes}}\circ$, then we write $\mathrm{Tor}_n^A(X, \circ)$ instead of the n-th projective derived functor, as usual. Thus $\mathrm{Tor}_n^A(X, Y)$, $n \in \mathbb{Z}$, are the homology spaces of the complex

$$X\widehat{\otimes}_A \mathcal{Q} : 0 \leftarrow X\widehat{\otimes}_A \mathcal{Q}_0 \overset{1\otimes_A d'}{\longleftarrow} X\widehat{\otimes}_A \mathcal{Q}_1 \longleftarrow \cdots,$$

where $(\mathcal{Q}, d')$ is a projective resolution of the left A-module Y. We set $F \perp X$ if $F_n(X) = \{0\}$, $n \geqslant 0$. If $F = X\underset{A}{\widehat{\otimes}}\circ$ then we write $X \perp_A Y$ (see [26]) for $Y \in A$-mod whenever $F \perp Y$. In this case we say that the modules X and Y are in the *transversality relation.*

LEMMA 1.1. *Let $X \in$ mod-A, $Y \in A$-mod. Then $X \perp_A Y$ if and only if* $(\circ\underset{A}{\widehat{\otimes}}Y) \perp X$.

Proof. Put $F = \circ \widehat{\underset{A}{\otimes}} Y$ which is a functor from mod-A into the category of the Fréchet spaces. By definition, $F \perp X$ means that $F_n(X) = \{0\}$ for all $n \in \mathbb{Z}_+$. Up to an algebraic isomorphism $F_n(X) = \mathrm{Tor}_n^A(X, Y)$ thanks to 3.4.26 of [20]. It remains to note that $X \perp_A Y$ means that $\mathrm{Tor}_n^A(X, Y) = \{0\}$ for all $n \in \mathbb{Z}_+$. ∎

In order to have a wider scope of applications, we introduce a *pseudoresolution* generalizing the resolution concept, which we shall use later. Let $X \in A$-mod and let $0 \leftarrow X \xleftarrow{\varepsilon} (\mathcal{P}, d)$ be an exact (not necessarily, admissible) sequence, where $(\mathcal{P}, d)$ is a complex in $\underline{A\text{-}\mathrm{mod}}$ of projective (respectively, free) modules with $\mathcal{P}_n = \{0\}$ for $n < 0$. In this case we say that $(\mathcal{P}, d)$ is a *projective* (respectively, *free*) *pseudoresolution of* X. A pseudoresolution of a left A-module X is called a *projective* (respectively, *free*) Tor-*resolution* if for each projective resolution $(\mathcal{Q}, d')$ of X the comparison mapping $\mathcal{P} \to \mathcal{Q}$ should induce isomorphisms

$$H_n(Y \widehat{\otimes}_A \mathcal{P}) \to H_n(Y \widehat{\otimes}_A \mathcal{Q}) = \mathrm{Tor}_n^A(Y, X), \quad n \in \mathbb{Z}_+$$

for each module $Y \in$ mod-A. Note that if $A = \mathcal{O}(\Omega)$ is the Fréchet algebra of all holomorphic functions on a Stein domain Ω in $\mathbb{C}^m$ then the Koszul complex of the A-bimodule $\mathcal{O}(\Omega \times \Omega)$ is a pseudoresolution of the bimodule $\mathcal{O}(\Omega)$ ([26], Proposition 4.3) having the length m, which is (admissible) free resolution if Ω is a polydomain. Using the nuclearity argument as in Propositions 4.5 and 2.8 of [25], we can derive that if $X \in A$-mod then the Koszul complex of this module is a free Tor-resolution.

1.2. LIE ALGEBRAS AND TAYLOR SPECTRUM. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The space of all Lie characters of a Lie algebra $\mathfrak{g}$ is denoted by $\Delta(\mathfrak{g})$. The universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{g}$ is denoted by $\mathcal{U}(\mathfrak{g})$. The space of all characters of $\mathcal{U}(\mathfrak{g})$ is identified with $\Delta(\mathfrak{g})$, that is, each Lie character $\lambda \in \Delta(\mathfrak{g})$ has unique extension to a character on $\mathcal{U}(\mathfrak{g})$ denoted by λ too. Take a basis $e = (e_1, \dots, e_n)$ in $\mathfrak{g}$. For an n-tuple $J = (j_1, \dots, j_n) \in \mathbb{Z}_+^n$ of nonnegative integers we put $e^J = e_1^{j_1} \cdots e_n^{j_n}$ to indicate the ordered monomial in $\mathcal{U}(\mathfrak{g})$ taken by the basis e. By Poincaré–Birkhoff–Witt theorem (see 2.2.1 of [4]), the set $\{e^J\} \subseteq \mathcal{U}(\mathfrak{g})$ of all ordered monomials is an algebraic basis in $\mathcal{U}(\mathfrak{g})$.

Now let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra with its vanishing lower central series $\{\mathfrak{g}^{(s)} : s \geqslant 1\}$, $\mathfrak{g}^{(1)} = \mathfrak{g}$, $\mathfrak{g}^{(s)} = [\mathfrak{g}, \mathfrak{g}^{(s-1)}]$, $s > 1$. A basis $e = (e_1, \dots, e_n)$ in $\mathfrak{g}$ is said to be a *triangular basis* if it obeys to the lower central series. Thus $[e_i, e_j] = \sum\limits_{k>j} c_{ij}^k e_k$ whenever $i < j$. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_c$ is graded with the positive integers $1, \dots, c$ then each basis subordinated to the latter decomposition (called a *graded basis*) is a triangular one. For a triangular basis $e = (e_1, \dots, e_n)$ of a nilpotent Lie algebra $\mathfrak{g}$, $e_{\mathrm{r}} = (e_{m+1}, \dots, e_n)$ will be a basis in $[\mathfrak{g}, \mathfrak{g}]$ for a certain m. We say that e_{r} is a *radical part* of e and $e_{\mathrm{s}} = (e_1, \dots, e_m)$ is a *semisimple part* of e. In that context, we also write $J_{\mathrm{s}} = (j_1, \dots, j_m)$ and $J_{\mathrm{r}} = (j_{m+1}, \dots, j_n)$ if $J = (j_1, \dots, j_n) \in \mathbb{Z}_+^n$, note also that $J = J_{\mathrm{s}} \cup J_{\mathrm{r}}$.

Let A be a Banach algebra and let $\mathfrak{g}$ be its finite-dimensional nilpotent Lie subalgebra. The closed associative envelope B of $\mathfrak{g}$ in A is a commutative algebra modulo its Jacobson radical $\operatorname{Rad} B$ thanks to Turovskii lemma [27]. Therefore $\operatorname{Rad} B$ is the set of all quasinilpotent elements in B which is just the left (or right) closed ideal in B generated by the Lie ideal $[\mathfrak{g},\mathfrak{g}]$. We say that $\mathfrak{g}$ is a *supernilpotent Lie algebra* in A if each $a \in [\mathfrak{g},\mathfrak{g}]$ is nilpotent in A. Note that if A is a finite-dimensional Banach algebra then each of its nilpotent Lie subalgebras is supernilpotent. Moreover, a commutative Lie subalgebra of a Banach algebra A is supernilpotent too. If $A = \mathcal{L}(X)$ is the Banach algebra of all bounded linear operators on a Banach space X and $\mathfrak{g}$ is a supernilpotent Lie algebra in A then we say that $\mathfrak{g}$ is a *supernilpotent Lie algebra of operators* [8], [5].

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, X a Fréchet space and let $\alpha : \mathfrak{g} \to \mathcal{L}(X)$ be a Lie representation, that is, X is a Fréchet $\mathfrak{g}$-module. The following complex

$$0 \leftarrow X \xleftarrow{d_0} X \otimes \mathfrak{g} \xleftarrow{d_1} \cdots \xleftarrow{d_{p-1}} X \otimes \bigwedge^{p} \mathfrak{g} \xleftarrow{d_p} \cdots$$

is called the *Koszul complex* of the pair (X, α), where

$$d_{p-1}(x \otimes \underline{u}) = \sum_{i=1}^{p} (-1)^{i+1} \alpha(u_i) x \otimes \underline{u}_i + \sum_{i<j} (-1)^{i+j-1} x \otimes [u_i, u_j] \wedge \underline{u}_{ij},$$

$\underline{u} = u_1 \wedge \cdots \wedge u_p \in \bigwedge^{p} \mathfrak{g}$, and it is denoted by $\operatorname{Kos}(X, \alpha)$. Obviously, $\alpha - \lambda : \mathfrak{g} \to \mathcal{L}(X)$ is a Lie representation for each $\lambda \in \Delta(\mathfrak{g})$. Recall [17], [13] that the *Taylor spectrum* $\sigma(\mathfrak{g}, X)$ of a $\mathfrak{g}$-module X is defined as the set of those $\lambda \in \Delta(\mathfrak{g})$ such that the Koszul complex $\operatorname{Kos}(X, \alpha - \lambda)$ fails to be exact.

2. THE COMPLEX DOMINATING OVER A MODULE

In this section we present Taylor–Helemskii–Putinar framework of the functional calculus, which has purely homological nature (see [26], Chapter 6 of [20], Chapter 5 of [16]).

Fix a Fréchet algebra A and a *finitely-projective* (respectively, *finitely-free*) *module* $X \in A$-mod, that is, X has a finite projective (respectively, free) Tor-resolution (see Subsection 1.1) say $\mathcal{P} = \{P^{-k}, \delta^{-k} : 0 \leqslant k \leqslant n\}$. In particular, the complex

$$0 \to P^{-n} \xrightarrow{\delta^{-n}} \cdots \longrightarrow P^{-2} \xrightarrow{\delta^{-2}} P^{-1} \xrightarrow{\delta^{-1}} P^{0} \xrightarrow{\varepsilon} X \to 0$$

is exact. For convenience, we are using the cochain form of the resolution. For a cochain complex $\mathcal{Y} = \{Y^k, d^k, k \in \mathbb{Z}_+\} \in \overline{\text{mod-}A}$, the following bicomplex

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & & \uparrow & \\
\cdots \longrightarrow & Y^{s-1}\widehat{\underset{A}{\otimes}}P^{-k+1} & \xrightarrow{d^{s-1}\otimes_A 1} & Y^{s}\widehat{\underset{A}{\otimes}}P^{-k+1} & \xrightarrow{d^{s}\otimes_A 1} & Y^{s+1}\widehat{\underset{A}{\otimes}}P^{-k+1} & \longrightarrow \cdots \\
 & \uparrow 1\otimes_A\delta^{-k} & & \uparrow 1\otimes_A\delta^{-k} & & \uparrow 1\otimes_A\delta^{-k} & \\
\cdots \longrightarrow & Y^{s-1}\widehat{\underset{A}{\otimes}}P^{-k} & \xrightarrow{d^{s-1}\otimes_A 1} & Y^{s}\widehat{\underset{A}{\otimes}}P^{-k} & \xrightarrow{d^{s}\otimes_A 1} & Y^{s+1}\widehat{\underset{A}{\otimes}}P^{-k} & \longrightarrow \cdots \\
 & \uparrow 1\otimes_A\delta^{-k-1} & & \uparrow 1\otimes_A\delta^{-k-1} & & \uparrow 1\otimes_A\delta^{-k-1} & \\
\cdots \longrightarrow & Y^{s-1}\widehat{\underset{A}{\otimes}}P^{-k-1} & \xrightarrow{d^{s-1}\otimes_A 1} & Y^{s}\widehat{\underset{A}{\otimes}}P^{-k-1} & \xrightarrow{d^{s}\otimes_A 1} & Y^{s+1}\widehat{\underset{A}{\otimes}}P^{-k-1} & \longrightarrow \cdots \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & \vdots & & \vdots & & \vdots &
\end{array}$$

is denoted by $\overline{\mathcal{Y}\widehat{\underset{A}{\otimes}}\mathcal{P}}$, and let $\mathcal{Y}\widehat{\underset{A}{\otimes}}\mathcal{P}$ be its total complex. By its very definition, the latter is the following complex:

$$0 \to Z^{-n} \xrightarrow{T^{-n}} Z^{-n+1} \xrightarrow{T^{-n+1}} \cdots \xrightarrow{T^{-1}} Z^0 \xrightarrow{T^0} Z^1 \xrightarrow{T^1} \cdots,$$

where

$$Z^m = \bigoplus_{s-k=m} Y^s\widehat{\underset{A}{\otimes}}P^{-k} \quad \text{and} \quad T^m = 1 \otimes_A \delta^{-k} + (-1)^s d^s \otimes_A 1 \quad \text{on} \quad Y^s\widehat{\underset{A}{\otimes}}P^{-k},$$

$m \geqslant -n$. Thus we have a functor

$$\circ\widehat{\underset{A}{\otimes}}\mathcal{P} : \overline{\text{mod-}A} \longrightarrow \overline{\mathbf{FS}}, \quad \mathcal{Y} \longmapsto \mathcal{Y}\widehat{\underset{A}{\otimes}}\mathcal{P}, \quad \overline{\varphi} \mapsto \overline{\varphi} \otimes_A 1_{\mathcal{P}}.$$

If H^k is the k-th homology functor on $\overline{\mathbf{FS}}$ then as in 6.2 of [20], we set $\operatorname{Tor}^k_A(\mathcal{Y}, X)$ for the composite functor $H^k(\circ\widehat{\underset{A}{\otimes}}\mathcal{P})$ applied to $\mathcal{Y}$, so $\operatorname{Tor}^k_A(\mathcal{Y}, X) = H^k(\mathcal{Y}\widehat{\underset{A}{\otimes}}\mathcal{P})$. Let us note that if we deal with the usual projective ($\mathbb{C}$-split) resolution $\mathcal{P}$ then it can be proved that $\operatorname{Tor}^k_A(\mathcal{Y}, X)$ does not depend upon the particular choice of $\mathcal{P}$. Further, if $\mathcal{Y}$ is reduced to a right A-module Y then $\operatorname{Tor}^k_A(\mathcal{Y}, X) = \operatorname{Tor}^A_k(Y, X)$, for $\mathcal{P}$ is a Tor-resolution (see Subsection 1.1). Whence $\operatorname{Tor}^0_A(A, X) = X$ and $\operatorname{Tor}^k_A(A, X) = 0$, $k \neq 0$ (in this case $\mathcal{Y}\widehat{\underset{A}{\otimes}}\mathcal{P} = \overline{\mathcal{Y}\widehat{\underset{A}{\otimes}}\mathcal{P}} = A\widehat{\underset{A}{\otimes}}\mathcal{P} = \mathcal{P}$).

Let $\mathcal{Y} = \{Y^k, d^k, k \in \mathbb{Z}_+\} \in \overline{\text{mod-}A}$. A morphism of the right A-modules $\eta : A \to Y^0$ such that $d^0\eta = 0$, is called an *augmentation of* $\mathcal{Y}$ and the pair $(\mathcal{Y}, \eta)$ is called an *augmented complex of the right A-modules.* The morphism of the augmented complexes is defined in the obvious way. Thus η defines a morphism of the complexes of the right A-modules $\overline{\eta} : A \to \mathcal{Y}$:

$$\begin{array}{ccc} \vdots & & \vdots \\ \uparrow & & \uparrow^{d^1} \\ 0 & \longrightarrow & Y^1 \\ \uparrow & & \uparrow^{d^0} \\ A & \xrightarrow{\eta} & Y^0 \\ \uparrow & & \uparrow \\ 0 & & 0 \end{array}$$

if $(\mathcal{Y}, \eta)$ is an augmented complex. The latter induces a morphism of the bicomplexes

$$\overline{\eta}\underset{A}{\widehat{\otimes}}\mathcal{P} : A\underset{A}{\widehat{\otimes}}\mathcal{P} \to \mathcal{Y}\underset{A}{\widehat{\otimes}}\mathcal{P}$$

so that the following diagram

$$\begin{array}{ccccccc} A\widehat{\otimes}_A P^0 & \longrightarrow & Y^0\widehat{\otimes}_A P^0 & \longrightarrow & Y^1\widehat{\otimes}_A P^0 & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ \vdots & & \vdots & & \vdots & & \\ \uparrow & & \uparrow & & \uparrow & & \\ A\widehat{\otimes}_A P^{-n} & \longrightarrow & Y^0\widehat{\otimes}_A P^{-n} & \longrightarrow & Y^1\widehat{\otimes}_A P^{-n} & \longrightarrow & \cdots \end{array}$$

is commutative. The latter in turn induces a morphism

$$H^k(\overline{\eta}\underset{A}{\widehat{\otimes}}\mathcal{P}) : \mathrm{Tor}_A^k(A, X) \to \mathrm{Tor}_A^k(\mathcal{Y}, X)$$

for every $k \in \mathbb{Z}$. But $H^k(\overline{\eta}\underset{A}{\widehat{\otimes}}\mathcal{P}) = 0$ for all $k \neq 0$ (since $\mathrm{Tor}_A^k(A, X) = 0$) and

$$\eta_* = H^0(\overline{\eta}\underset{A}{\widehat{\otimes}}\mathcal{P}) : X \to \mathrm{Tor}_A^0(\mathcal{Y}, X), \quad \eta_*(\varepsilon(z)) = (\eta(1_A) \otimes_A z)^\sim (\mathrm{mod}\,\mathrm{im}\, T^{-1}),$$

where $z \in P^0$, $\varepsilon : P^0 \to X$, $T^{-1} : Z^{-1} \to Z^0$ are the morphisms of the relevant complexes.

DEFINITION 2.1. An augmented complex of the right A-modules $(\mathcal{Y}, \eta)$ is said to be *dominating over* X, in this case we write $(\mathcal{Y}, \eta) \gg X$, if η_* is a topological isomorphism and $\mathrm{Tor}_A^k(\mathcal{Y}, X) = 0$ for all k, $k \neq 0$.

Let us assume that $\mathcal{Y}$ is reduced to a right A-module Y and $\eta : A \to Y$ is a right A-module morphism. By Definition 2.1, $(Y, \eta) \gg X$ means that $\mathrm{Tor}_k^A(Y, X) = 0$ for all k, $k \neq 0$, and the morphism $\eta_* : X \to \mathrm{Tor}_0^A(Y, X)$ is a topological isomorphism. Consider the following

$$\theta : \mathrm{Tor}_0^A(Y, X) \longrightarrow Y\underset{A}{\widehat{\otimes}}X, \quad \theta(y \otimes_A z)^\sim \,\mathrm{mod}(\mathrm{im}(1_Y \otimes_A \delta^{-1})) = y \otimes_A \varepsilon(z),$$

$y \in Y, z \in \mathcal{P}_0$, continuous linear mapping, which plays the key role in the forthcoming assertion.

LEMMA 2.2. *Let $\eta : A \to Y$ be a right A-module morphism and let $X \in A$-mod be a finitely-projective module. If the morphism $\widetilde{\eta} : X \to Y\underset{A}{\widehat{\otimes}}X$, $\widetilde{\eta}(x) = \eta(1_A) \otimes_A x$, is a topological isomorphism then $\mathrm{Tor}_0^A(Y, X) = \overline{\{0\}} \oplus X$ up to a topological isomorphism, where $\overline{\{0\}}$ is the subspace in $\mathrm{Tor}_0^A(Y, X)$ of all points adherent to zero. In particular, $(Y, \eta) \gg X$ if and only if $\mathrm{Tor}_k^A(Y, X) = 0$ for all k, $k \neq 0$, $\mathrm{Tor}_0^A(Y, X)$ is Hausdorff and $\widetilde{\eta} : X \to Y\underset{A}{\widehat{\otimes}}X$ is a topological isomorphism.*

Proof. By assumption we have the inverse continuous morphism $\widetilde{\eta}^{-1} : Y\underset{A}{\widehat{\otimes}}X \to X$. Consider the following diagram

$$
\begin{array}{ccccccccc}
0 \leftarrow & X & \overset{\varepsilon}{\longleftarrow} & & \mathcal{P}^0 & \overset{\delta^{-1}}{\longleftarrow} & \mathcal{P}^1 & \longleftarrow & \cdots \\
& \uparrow\downarrow^{\widetilde{\eta}} & & & \downarrow & & \downarrow & & \\
0 \leftarrow & Y\widehat{\otimes}_A X & \overset{1\otimes_A\varepsilon}{\longleftarrow} & & Y\widehat{\otimes}_A \mathcal{P}^0 & \overset{1\otimes_A\delta^{-1}}{\longleftarrow} & Y\widehat{\otimes}_A \mathcal{P}^1 & \longleftarrow & \cdots \\
& {}_{\theta}\nwarrow & & & \swarrow_{\pi} & & & & \\
& & \mathrm{Tor}_0^A(Y, X) & & & & & &
\end{array}
$$

which is commutative, where

$$\pi : Y\widehat{\otimes}_A \mathcal{P}^0 \to \mathrm{Tor}_0^A(Y, X) = Y\widehat{\otimes}_A \mathcal{P}^0 / \operatorname{im}(1 \otimes_A \delta^{-1})$$

is the quotient mapping. Note that θ is surjective. Indeed, take $\omega \in Y\widehat{\otimes}_A X$. Then $\widetilde{\eta}^{-1}(\omega) = \varepsilon(z)$ for a certain $z \in \mathcal{P}^0$. It follows that (see to the diagram)

$$\theta((\eta(1_A) \otimes_A z)^{\sim}) = \eta(1_A) \otimes_A \varepsilon(z) = (1 \otimes_A \varepsilon)(\eta(1_A) \otimes_A z) = \widetilde{\eta}(\varepsilon(z)) = \omega.$$

Consider the mapping $\tau = \widetilde{\eta}^{-1}\theta : \mathrm{Tor}_0^A(Y, X) \to X$. Then

$$
\begin{aligned}
\tau\eta_*(x) &= \tau\eta_*(\varepsilon(z)) = \tau(\eta(1_A) \otimes_A z)^{\sim} = \widetilde{\eta}^{-1}\theta(\eta(1_A) \otimes_A z)^{\sim} \\
&= \widetilde{\eta}^{-1}(\eta(1_A) \otimes_A \varepsilon(z)) = \widetilde{\eta}^{-1}\widetilde{\eta}(x) = x
\end{aligned}
$$

for all $x \in X$, that is, $\tau\eta_* = 1_X$ and $\eta_* : X \to \mathrm{Tor}_0^A(Y, X)$ is a topological isomorphism onto its range. Further, take $\omega \in Y\widehat{\otimes}_A \mathcal{P}^0$. Then $\theta(\omega^{\sim}) = \widetilde{\eta}(x)$ for some $x = \varepsilon(z) \in X$. Thereby $\theta(\omega^{\sim}) = \widetilde{\eta}(\varepsilon(z)) = (1 \otimes_A \varepsilon)(\eta(1_A) \otimes_A z) = \theta((\eta(1_A) \otimes_A z)^{\sim}) = \theta\eta_*\varepsilon(z) = \theta(\eta_* x)$ and $\omega^{\sim} - \eta_*(x) \in \ker(\theta)$. But it is known ([20], 3.4.26) that $\ker(\theta) = \overline{\{0\}}$. Thus we have a continuous projection $p : \mathrm{Tor}_0^A(Y, X) \to \mathrm{Tor}_0^A(Y, X)$, $p = \eta_*\tau$ onto $\eta_*(X)$, whose kernel is $\ker(p) = \ker(\theta) = \overline{\{0\}}$. Therefore $\mathrm{Tor}_0^A(Y, X) = \overline{\{0\}} \oplus \eta_*(X)$.

Finally, if $\mathrm{Tor}_0^A(Y, X)$ is Hausdorff then $\mathrm{Tor}_0^A(Y, X) = \eta_*(X)$ and the mapping $\eta_* : X \to \mathrm{Tor}_0^A(Y, X)$ turns out to be a topological isomorphism, that is, $(Y, \eta) \gg X$ if additionally $\mathrm{Tor}_k^A(Y, X) = 0$ for all k, $k \neq 0$. ∎

The following technical result from 6.2.25 of [20] will be used later.

LEMMA 2.3. *Let $X \in A$-mod be a module with a finite projective resolution $\mathcal{P}$, $(\mathcal{Y}, \eta)$ and $(\mathcal{Z}, \zeta)$ augmented complexes of right A-modules, $\overline{\varphi} : (\mathcal{Y}, \eta) \to (\mathcal{Z}, \zeta)$ a* *morphism and let $\overline{\mathcal{N}}$ be a Fréchet space bicomplex with bounded diagonals. Assume that one of the following sequences of Fréchet space bicomplexes*

$$0 \to \overline{\mathcal{N}} \longrightarrow \overline{\mathcal{Y}\underset{A}{\widehat{\otimes}}\mathcal{P}} \overset{\overline{\varphi\otimes_A 1}}{\longrightarrow} \overline{\mathcal{Z}\underset{A}{\widehat{\otimes}}\mathcal{P}} \to 0 \quad or \quad 0 \to \overline{\mathcal{Y}\underset{A}{\widehat{\otimes}}\mathcal{P}} \overset{\overline{\varphi\otimes_A 1}}{\longrightarrow} \overline{\mathcal{Z}\underset{A}{\widehat{\otimes}}\mathcal{P}} \longrightarrow \overline{\mathcal{N}} \to 0$$

is exact. If the total complex of $\overline{\mathcal{N}}$ is exact then $(\mathcal{Y}, \eta) \gg X$ and $(\mathcal{Z}, \zeta) \gg X$ are true or false simultaneously.

Proof. Assume the first one is exact. If $\mathcal{N}$ is the total complex of the bicomplex $\overline{\mathcal{N}}$ then we have the following

$$0 \to \mathcal{N} \longrightarrow \mathcal{Y}\underset{A}{\widehat{\otimes}}\mathcal{P} \overset{\varphi\otimes_A 1}{\longrightarrow} \mathcal{Z}\underset{A}{\widehat{\otimes}}\mathcal{P} \to 0$$

exact sequence of Fréchet space complexes. Since $\mathcal{N}$ is exact, it follows that the canonical mappings $\varphi_* : H^k(\mathcal{Y}\underset{A}{\widehat{\otimes}}\mathcal{P}) \to H^k(\mathcal{Z}\underset{A}{\widehat{\otimes}}\mathcal{P})$ are topological isomorphisms thanks to the open mapping theorem (for the details see 0.5.9 of [20]). Hence $\mathrm{Tor}^k_A(\mathcal{Y}, X) = \mathrm{Tor}^k_A(\mathcal{Z}, X)$ up to a topological isomorphism for each k. It remains to note (see Definition 2.1) that the following diagram is commutative and the mapping φ_* is a topological isomorphism:

$$\begin{tikzcd} \mathrm{Tor}^0_A(\mathcal{Y}, X) \arrow[rr, "\varphi_*"] & & \mathrm{Tor}^0_A(\mathcal{Z}, X) \\ & X \arrow[ul, "\eta_*"] \arrow[ur, "\zeta_*"'] & \end{tikzcd} .$$ ∎

Now let B be another algebra and let $\iota : A \to B$ be an algebra homomorphism. Then $B \in A\text{-mod-}A$ via the mapping ι, and using Lemma 2.2, we derive that $(B, \iota) \gg X$ means that $\mathrm{Tor}^A_k(B, X) = 0$ for all k, $k \neq 0$, $\mathrm{Tor}^A_0(B, X)$ is Hausdorff and $X = B\underset{A}{\widehat{\otimes}}X$ up to an isomorphism in **FS**. In particular, the A-module structure on X is naturally extended up to a B-module structure on X, thus $X \in B\text{-mod}$. This approach is generalized in the following assertion.

THEOREM 2.4. *Let $\iota : A \to B$ be an algebra homomorphism, $X \in A\text{-mod}$ a finitely projective module, and let $(\mathcal{Y}, \eta)$ be an augmented complex of right A-modules such that $\mathcal{Y} \in \overline{B\text{-}\mathrm{mod}\text{-}A}$, $(\mathcal{Y}, \eta) \gg X$, and $\overline{\eta} : A \to \mathcal{Y}$ is also morphism of left A-modules. Then X turns into a left B-module such that its A-module structure via ι is reduced to the original one.*

Proof. By assumption, $\mathcal{Y}\underset{A}{\widehat{\otimes}}\mathcal{P} \in \overline{B\text{-}\mathrm{mod}}$ and $\eta_* : X \to \mathrm{Tor}^0_A(\mathcal{Y}, X)$ is a topological isomorphism. Then $\mathrm{Tor}^0_A(\mathcal{Y}, X) = H^0(\mathcal{Y}\underset{A}{\widehat{\otimes}}\mathcal{P})$ is Hausdorff and we have $\mathrm{Tor}^0_A(\mathcal{Y}, X) \in B\text{-mod}$. Undoubtedly, $\mathcal{Y}\underset{A}{\widehat{\otimes}}\mathcal{P}$ is a complex of left A-modules (via ι) too. Take $x \in X$. Then $x = \varepsilon(z)$ for a certain $z \in \mathcal{P}^0$. Moreover,

$$\begin{aligned}\eta_*(ax) &= \eta_*(a\varepsilon(z)) = \eta_*(\varepsilon(az)) = (\eta(1_A) \otimes_A az)^\sim (\mathrm{mod}\,\mathrm{im}\,T^{-1}) \\ &= (\eta(1_A)a \otimes_A z)^\sim (\mathrm{mod}\,\mathrm{im}\,T^{-1}) = (\eta(a) \otimes_A z)^\sim (\mathrm{mod}\,\mathrm{im}\,T^{-1})\end{aligned}$$

$$= (\iota(a)\eta(1_A) \otimes_A z)^{\sim}(\operatorname{mod} \operatorname{im} T^{-1})$$
$$= \iota(a)((\eta(1_A) \otimes_A z)^{\sim}(\operatorname{mod} \operatorname{im} T^{-1})) = \iota(a)\eta_*(x)$$

for every $a \in A$. The latter means that η_* is a morphism of left A-modules. Thus X is identified with $\operatorname{Tor}_A^0(\mathcal{Y}, X)$ as a left A-module. Now we set $b \cdot x = \eta_*^{-1}(b\eta_*(x))$ for $b \in B$ and $x \in X$. Then $X \in B$-mod and $\iota(a) \cdot x = \eta_*^{-1}(\iota(a)\eta_*(x)) = \eta_*^{-1}(\eta_*(ax)) = ax$ for $a \in A$. ∎

3. FRÉCHET SHEAVES

In this section we review briefly the Fréchet sheaves and their cohomologies, that will be used later.

Let Ω be a Hausdorff topological space. By a *Fréchet* (or more generally *locally convex space*) *presheaf over* Ω we mean a contravariant functor $\mathcal{S}$ from the category of all open subsets in Ω into the category **FS**. Thus $\mathcal{S}(U) \in$ **FS** whenever $U \subseteq \Omega$ is open, and all restriction mappings $\mathcal{S}(U) \to \mathcal{S}(V)$, $V \subseteq U$, are morphisms in **FS**. A Fréchet presheaf $\mathcal{S}$ over Ω is said to be a *Fréchet sheaf* if $\mathcal{S}$ possesses the sheaf axioms additionally ([19], 2.1.1) . By analogy is defined a Fréchet algebra sheaf. The cohomologies of the space Ω with coefficients in a Fréchet sheaf $\mathcal{S}$ (see 2.4 of [19]) are denoted by $H^n(\Omega, \mathcal{S})$, $n \in \mathbb{Z}_+$. We say that a topological space Ω is $\mathcal{S}$*-acyclic* if $H^n(\Omega, \mathcal{S}) = \{0\}$ for all $n \in \mathbb{N}$. An open subset $U \subseteq \Omega$ is said to be $\mathcal{S}$-acyclic if the topological space U (equipped with the topology from Ω) is $\mathcal{S}|U$-acyclic, where $\mathcal{S}|U$ is the sheaf over U induced by $\mathcal{S}$. Finally, a Fréchet sheaf $\mathcal{S}$ over Ω is said to be a *nuclear sheaf* if $\mathcal{S}(U)$ is a nuclear space whenever U is an open $\mathcal{S}$-acyclic subset in Ω. In the same way is defined a *nuclear Fréchet algebra sheaf.*

The sheaf of germs of holomorphic functions over a complex space $\Omega = \mathbb{C}^n$ is denoted by $\mathcal{O}$. Fix a Fréchet space X. The assignment $U \mapsto \mathcal{O}(U, X)$ defines a sheaf over $\mathbb{C}^n$ where U runs over all open subsets in $\mathbb{C}^n$. The latter is the sheaf of germs of X-valued holomorphic functions denoted by $\mathcal{O}\widehat{\otimes}X$. Actually, $\mathcal{O}\widehat{\otimes}X$ is the topological tensor product of the sheaf $\mathcal{O}$ and the constant sheaf generated by the space X. The following result is well known ([21], 4.2.6, and [16], Appendix 1).

PROPOSITION 3.1. *Let U be a pseudoconvex open set in $\mathbb{C}^n$, X a Fréchet space and let $C^\infty_{0,p}(U, X)$ be the space of all differential forms of bidegree $(0, p)$ with coefficients in the space $C^\infty(U, X)$ of all X-valued C^∞-function on U. Then the $\overline{\partial}$-sequence is exact:*

$$0 \to \mathcal{O}(U, X) \longrightarrow C^\infty_{0,0}(U, X) \xrightarrow{\overline{\partial}} C^\infty_{0,1}(U, X) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} C^\infty_{0,n}(U, X) \to 0\,.$$

The following corollary of this result will be used later.

COROLLARY 3.2. *Let X be a Fréchet space and let $C^\infty_{0,p}\widehat{\otimes}X$ be the Fréchet sheaf of germs of X-valued exterior differential forms of bidegree $(0, p)$ over $\mathbb{C}^n$. All sheaves*

$\mathcal{C}^{\infty}_{0,p}\widehat{\otimes}X$, $0 \leqslant p \leqslant n$, are soft and the $\overline{\partial}$-sequence

$$0 \to \mathcal{O}\widehat{\otimes}X \longrightarrow \mathcal{C}^{\infty}_{0,0}\widehat{\otimes}X \xrightarrow{\overline{\partial}} \mathcal{C}^{\infty}_{0,1}\widehat{\otimes}X \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{C}^{\infty}_{0,n}\widehat{\otimes}X \to 0$$

is an exact sequence of Fréchet sheaves. Thereby, an open subset $U \subseteq \mathbb{C}^n$ is $\mathcal{O}\widehat{\otimes}X$-acyclic whenever U is pseudoconvex.

Proof. First, note that there are canonical identifications

$$C^{\infty}_{0,p}(U,X) = C^{\infty}_{0,p}(U)\widehat{\otimes}X,$$

where $U \subseteq \mathbb{C}^n$ is open. Hence, the sheaf $\mathcal{C}^{\infty}_{0,p}\widehat{\otimes}X$ is completely defined by the assignment $U \mapsto C^{\infty}_{0,p}(U,X)$. Let $\mathcal{C}^{\infty}$ be the sheaf of germs of complex C^{∞}-functions over $\mathbb{C}^n$. Then $\mathcal{C}^{\infty}$ is a soft sheaf of algebras (see Chapter 2, item 3.7 of [19]). It follows that all sheaves $\mathcal{C}^{\infty}_{0,p}\widehat{\otimes}X$ as left $\mathcal{C}^{\infty}$-modules have to be soft too ([19], Chapter 2, Theorem 3.7.1). The exactness of the asserted sheaf complex immediately follows from Proposition 3.1. Thus we have a soft resolution of the sheaf $\mathcal{O}\widehat{\otimes}X$, whence cohomologies $H^n(U,\mathcal{O}\widehat{\otimes}X)$, $n \in \mathbb{Z}_+$, may be calculated by means of this resolution ([19], Chapter 2, Theorem 4.7.1). Appealing again to Proposition 3.1, we conclude that $H^n(U,\mathcal{O}\widehat{\otimes}X) = \{0\}$, $n \in \mathbb{N}$, i.e. U is $\mathcal{O}\widehat{\otimes}X$-acyclic, whenever U is pseudoconvex. ∎

3.1. COHOMOLOGIES OF AN OPEN COVER. Now let us remind cohomologies of an open cover of the space Ω. Take a locally convex space (or briefly, l.c.s.) presheaf $\mathcal{S}$ and an open cover $\mathfrak{V} = \{V_i\}_{i\in I}$ of Ω. We set $V_\alpha = V_{i_0} \cap \cdots \cap V_{i_n}$ for a $(n+1)$-tuple $\alpha = (i_0,\ldots,i_n) \in I^{n+1}$. Fix $n \in \mathbb{Z}_+$. Let us define an l.c.s. presheaf of Čech n-cochains $\mathcal{C}^n(\mathfrak{V},\mathcal{S})$ over Ω. For an open set $V \subseteq \Omega$, let $\mathcal{C}^n(\mathfrak{V},\mathcal{S})(V) = \prod_{\alpha\in I^{n+1}} \mathcal{S}(V_\alpha \cap V)$, where $\mathcal{S}(V_\alpha \cap V)$ is the l.c.s. space of all sections of $\mathcal{S}$ over $V_\alpha \cap V$. If $W \subseteq V$, the restriction mapping of the sheaf $\mathcal{S}$ clearly defines a morphism

$$\mathcal{C}^n(\mathfrak{V},\mathcal{S})(V) \to \mathcal{C}^n(\mathfrak{V},\mathcal{S})(W), \quad f \mapsto f|_W,$$

where $f|_W(\alpha) = f(\alpha)|_{V_\alpha\cap W}$, $\alpha \in I^{n+1}$. The space $\mathcal{C}^n(\mathfrak{V},\mathcal{S})(V)$ equipped with the direct product topology is an l.c.s. space, and $\mathcal{C}^n(\mathfrak{V},\mathcal{S})$ is an l.c.s. presheaf over Ω. Moreover, $\mathcal{C}^n(\mathfrak{V},\mathcal{S})$ is an l.c.s. sheaf if so is $\mathcal{S}$, and $\mathcal{C}^n(\mathfrak{V},\mathcal{S})$ is a Fréchet sheaf whenever $\mathcal{S}$ is a Fréchet sheaf and $\mathfrak{V}$ is a countable cover. Finally, $\mathcal{C}^n(\mathfrak{V},\mathcal{S})(\Omega)$ is a nuclear space if $\mathcal{S}$ is a nuclear sheaf and all V_α are $\mathcal{S}$-acyclic. Let $V \subseteq \Omega$ be an open subset. We define a continuous linear mapping

$$\delta^n_{\mathfrak{V}}(V) : \mathcal{C}^n(\mathfrak{V},\mathcal{S})(V) \to \mathcal{C}^{n+1}(\mathfrak{V},\mathcal{S})(V), \quad (\delta^n_{\mathfrak{V}}(V)f)(\alpha) = \sum_{j=0}^{n+1}(-1)^j f(\alpha_j)|_{V_\alpha\cap V},$$

where $\alpha = (i_0,\ldots,i_{n+1}) \in I^{n+2}$, $\alpha_j = (i_0,\ldots,\widehat{i_j},\ldots,i_{n+1})$ is the result of throwing out i_j from α. One can easily check that $\delta^n_{\mathfrak{V}} = \{\delta^n_{\mathfrak{V}}(V)\}$ implements a morphism

$\delta^n_{\mathfrak{V}} : \mathcal{C}^n(\mathfrak{V}, \mathcal{S}) \to \mathcal{C}^{n+1}(\mathfrak{V}, \mathcal{S})$ of l.c.s. presheaves. Moreover, the mapping

$$\mathcal{S}(V) \to \prod_{i \in I} \mathcal{S}(V_i \cap V), \quad a \mapsto \overline{a}, \quad \overline{a}(i) = a|_{V_i \cap V},$$

defines an l.c.s. presheaf morphism $\varepsilon_{\mathfrak{V}} : \mathcal{S} \to \mathcal{C}^0(\mathfrak{V}, \mathcal{S})$, and the sequence

$$\mathcal{C}^\bullet(\mathfrak{V}, \mathcal{S}) = \{\mathcal{C}^n(\mathfrak{V}, \mathcal{S}), \delta^n_{\mathfrak{V}}, n \in \mathbb{Z}_+\}$$

is a chain complex of presheaves (see Chapter 2, item 5.2 of [19]) augmented by the morphism $\varepsilon_{\mathfrak{V}}$. The sequence $\mathcal{C}^\bullet(\mathfrak{V}, \mathcal{S})$ is called a *Čech complex of the presheaf $\mathcal{S}$ with respect to the cover $\mathfrak{V}$*. The following result is well known (see for instance Chapter 2, Theorem 5.2.1 of [19]).

THEOREM 3.3. *Let $\mathfrak{V}$ be an open cover of Ω and $\mathcal{S}$ is a* l.c.s. *sheaf over Ω. The following sequence is an exact complex of sheaves, that is, $\mathcal{C}^\bullet(\mathfrak{V}, \mathcal{S})$ is a resolution of the sheaf $\mathcal{S}$:*

$$0 \to \mathcal{S} \xrightarrow{\varepsilon_{\mathfrak{V}}} \mathcal{C}^0(\mathfrak{V}, \mathcal{S}) \xrightarrow{\delta^0_{\mathfrak{V}}} \mathcal{C}^1(\mathfrak{V}, \mathcal{S}) \xrightarrow{\delta^1_{\mathfrak{V}}} \cdots \xrightarrow{\delta^{n-1}_{\mathfrak{V}}} \mathcal{C}^n(\mathfrak{V}, \mathcal{S}) \xrightarrow{\delta^n_{\mathfrak{V}}} \cdots$$

The cohomology of the complex of the global sections

$$\mathcal{C}^\bullet(\mathfrak{V}, \mathcal{S})(\Omega) = \{\mathcal{C}^n(\mathfrak{V}, \mathcal{S})(\Omega), \delta^n_{\mathfrak{V}}(\Omega), n \in \mathbb{Z}_+\}$$

is called the *cohomology of the cover* $\mathfrak{V}$ with coefficients in the sheaf $\mathcal{S}$ and they are denoted by $H^n(\mathfrak{V}, \mathcal{S})$, $n \in \mathbb{Z}_+$. One can easily check that $H^0(\mathfrak{V}, \mathcal{S}) = \mathcal{S}(\Omega) = H^0(\Omega, \mathcal{S})$.

4. THE DOMINATING ČECH COMPLEX

In this section we present some modification of Taylor–Helemskii–Putinar framework (Section 2) to demonstrate how does spectrum relate to the functional calculus problem. First, we introduce the *Putinar spectrum* of a module over an algebra of the global sections of a sheaf.

4.1. PUTINAR SPECTRUM. Let $\mathcal{S}$ be an l.c.s. sheaf over a topological space Ω. We say that Ω is a *locally $\mathcal{S}$-acyclic space* if each point in Ω has a fundamental system of $\mathcal{S}$-acyclic open neighborhoods.

EXAMPLE 4.1. Let $\Omega = \mathbb{C}^n$, X a Fréchet space and let $\mathcal{S} = \mathcal{O}\widehat{\otimes}X$ be the sheaf of germs of X-valued holomorphic functions over $\mathbb{C}^n$. Since each point in the complex space $\mathbb{C}^n$ has a fundamental system of convex neighborhoods, it follows that $\mathbb{C}^n$ is a locally $\mathcal{O}\widehat{\otimes}X$-acyclic space thanks to Corollary 3.2.

The following concept of a spectrum plays a central role in our paper.

DEFINITION 4.2. Let Ω be a locally $\mathcal{S}$-acyclic space, where $\mathcal{S}$ is a Fréchet algebra sheaf over Ω, $A = \mathcal{S}(\Omega)$ and let $X \in A$-mod be a Fréchet module. The resolvent set $\operatorname{res}(\mathcal{S}, X)$ of the A-module X with respect to the sheaf $\mathcal{S}$ is defined as a subset of all points $\lambda \in \Omega$ such that there exists an open neighborhood D_λ of λ with $\mathcal{S}(W) \perp_A X$ for each open $\mathcal{S}$-acyclic subset $W \subseteq D_\lambda$. We set

$$\sigma(\mathcal{S}, X) = \Omega \setminus \operatorname{res}(\mathcal{S}, X).$$

This set is called the *Putinar spectrum* of the A-module X with respect to the sheaf $\mathcal{S}$.

As immediate from Definition 4.2, $\operatorname{res}(\mathcal{S}, X)$ is an open set in Ω, therefore the Putinar spectrum $\sigma(\mathcal{S}, X)$ is closed.

4.2. THE MORPHISM OF ČECH COMPLEXES. Now let us assume that $\mathcal{S}$ is an l.c.s. presheaf over a topological space Ω, U an open subset of Ω, and let $\mathfrak{V} = \{V_i\}_{i\in I}$ be an infinite open cover of Ω such that $U = \bigcup\limits_{i\in J} V_i$ for some subset $J \subseteq I$. Assume that there is a bijection mapping $\varphi : I \to J$ additionally, and let $\mathfrak{U} = \{U_i\}_{i\in I}$, where $U_i = V_{\varphi(i)}$. Fix $n \in \mathbb{Z}_+$. For an open subset $V \subseteq \Omega$, let us define a continuous epimorphism

$$\varphi_{\mathcal{S}}^{(n)}(V) : \mathcal{C}^n(\mathfrak{V}, \mathcal{S})(V) \to \mathcal{C}^n(\mathfrak{U}, \mathcal{S}|U)(V \cap U), \quad (\varphi_{\mathcal{S}}^{(n)}(V)f)(\alpha) = f(\varphi^{(n)}(\alpha)),$$

where $\alpha = (i_0, \ldots, i_n) \in I^{n+1}$, $\varphi^{(n)}(\alpha) = (\varphi(i_0), \ldots, \varphi(i_n)) \in J^{n+1}$. One easily check that $\ker(\varphi_{\mathcal{S}}^{(n)}(V)) = \prod\limits_{\alpha\in M_n} \mathcal{S}(V_\alpha \cap V)$, where M_n is a subset of I^{n+1} comprising all tuples $\alpha = (i_0, \ldots, i_n)$ such that at least one $i_s \in I \setminus J$. Note that for a such taken $\alpha \in M_n$, $V_\alpha \subseteq V_i$ for a certain $i \in I \setminus J$. Moreover, the family $\varphi_{\mathcal{S}} = \{\varphi_{\mathcal{S}}^{(n)}(V)\}$ is compatible with the restriction mappings and therefore it defines a presheaf cohomomorphism $\mathcal{C}^n(\mathfrak{V}, \mathcal{S}) \to \mathcal{C}^n(\mathfrak{U}, \mathcal{S}|U)$ ([2], Chapter 1, Section 4). Take $f \in \mathcal{C}^n(\mathfrak{V}, \mathcal{S})(V)$, $g = \delta^n_{\mathfrak{U}}(V \cap U)\varphi_{\mathcal{S}}^{(n)}(V)f$. Then

$$\begin{aligned} g(\alpha) &= \sum_{j=0}^{n+1} (-1)^j (\varphi_{\mathcal{S}}^{(n)}(V)f)(\alpha_j)|_{U_\alpha \cap V} = \sum_{j=0}^{n+1} (-1)^j f(\varphi^{(n)}(\alpha_j))|_{U_\alpha \cap V} \\ &= \sum_{j=0}^{n+1} (-1)^j f(\varphi^{(n+1)}(\alpha)_j)|_{V_{\varphi^{(n+1)}(\alpha)} \cap V} = (\delta^n_{\mathfrak{V}}(V)f)(\varphi^{(n+1)}(\alpha)) \\ &= (\varphi_{\mathcal{S}}^{(n+1)}(V)\delta^n_{\mathfrak{V}}(V)f)(\alpha), \end{aligned}$$

where $\alpha \in I^{n+2}$. It follows that the family $\varphi_{\mathcal{S}}$ implements a morphism $\overline{\varphi}_{\mathcal{S}} : \mathcal{C}^\bullet(\mathfrak{V}, \mathcal{S}) \to \mathcal{C}^\bullet(\mathfrak{U}, \mathcal{S}|U)$ of presheaf complexes. Note also that

$$\varphi_{\mathcal{S}}^{(0)}(V)(\varepsilon_{\mathfrak{V}}(V)f)(i) = (\varepsilon_{\mathfrak{V}}(V)f)(\varphi(i)) = f|_{V_{\varphi(i)} \cap V} = f|_{U_i \cap V} = (\varepsilon_{\mathfrak{U}}(V \cap U)f)(i)$$

for all $i \in I$. It follows that $\overline{\varphi}_{\mathcal{S}}$ is a morphism of the augmented complexes.

Now let $\mathcal{S}$ be a Fréchet algebra presheaf, $A = \mathcal{S}(\Omega)$, $B = \mathcal{S}(U)$ and let $\mathcal{C}^\bullet(\mathfrak{V}) = \mathcal{C}^\bullet(\mathfrak{V}, \mathcal{S})(\Omega)$, $\mathcal{C}^\bullet(\mathfrak{U}) = \mathcal{C}^\bullet(\mathfrak{U}, \mathcal{S}|U)(U)$, $\varepsilon = \varepsilon_{\mathfrak{V}}(\Omega)$. Obviously, A and B are Fréchet algebras, $\mathcal{C}^\bullet(\mathfrak{V}), \mathcal{C}^\bullet(\mathfrak{U}) \in \overline{\operatorname{mod}\text{-}A}$. Moreover, $\mathcal{C}^\bullet(\mathfrak{U}) \in \overline{B\text{-}\operatorname{mod}\text{-}A}$, for $\mathcal{S}(U_\alpha) \in B$-mod-$A$ via the restriction homomorphisms $\mathcal{S}(U) \to \mathcal{S}(U_\alpha)$ and $\mathcal{S}(\Omega) \to \mathcal{S}(U_\alpha)$. Note also that

$$\overline{\varphi} = \overline{\varphi}_{\mathcal{S}}(\Omega) : \mathcal{C}^\bullet(\mathfrak{V}) \to \mathcal{C}^\bullet(\mathfrak{U})$$

is a right A-module epimorphism of the relevant complexes. Further, let $X \in A$-mod be a finitely-free Fréchet module with its finite A-free Tor-resolution $\mathcal{P} = \{P^{-k}, \delta^{-k} : 0 \leqslant k \leqslant n\}$ of length n, so, each $P^{-k} = A\widehat{\otimes}E_k$ for some Fréchet spaces E_k. If $Y \in$ mod-A is a Fréchet module then we have the following complex

$$Y\underset{A}{\widehat{\otimes}}\mathcal{P} = \{Y\widehat{\otimes}E_k, 1_Y\widehat{\otimes}_A\delta^{-k} : 0 \leqslant k \leqslant n\}.$$

Thus $Y \perp_A X$ if and only if $Y\underset{A}{\widehat{\otimes}}\mathcal{P}$ is exact.

The mapping $\overline{\varphi}$ induces a bicomplex epimorphism (we deal with the category **FS**)

$$\overline{\varphi}\widehat{\otimes}_A 1 : \overline{\mathcal{C}^\bullet(\mathcal{V})\underset{A}{\widehat{\otimes}}\mathcal{P}} \to \overline{\mathcal{C}^\bullet(\mathcal{U})\underset{A}{\widehat{\otimes}}\mathcal{P}}$$

(see 2.4.12 of [20]). Let $\overline{\mathcal{N}}$ be its kernel. Note that the n-th column of the bicomplex $\overline{\mathcal{C}^\bullet(\mathcal{V})\underset{A}{\widehat{\otimes}}\mathcal{P}}$ (respectively, $\overline{\mathcal{C}^\bullet(\mathcal{U})\underset{A}{\widehat{\otimes}}\mathcal{P}}$) is the complex $\mathcal{C}^n(\mathcal{V})\underset{A}{\widehat{\otimes}}\mathcal{P}$ (respectively, $\mathcal{C}^n(\mathcal{U})\underset{A}{\widehat{\otimes}}\mathcal{P}$). Moreover, the mapping

$$\overline{\varphi}\widehat{\otimes}_A 1 : \mathcal{C}^n(\mathcal{V})\widehat{\otimes}E_k \to \mathcal{C}^n(\mathcal{U})\widehat{\otimes}E_k$$

is reduced to the epimorphism $\varphi^{(n)}_{\mathcal{S}_k}(\Omega) : \mathcal{C}^n(\mathcal{V}, \mathcal{S}_k)(\Omega) \to \mathcal{C}^n(\mathcal{U}, \mathcal{S}_k|U)(U)$, where $\mathcal{S}_k = \mathcal{S}\widehat{\otimes}E_k$ is a Fréchet presheaf over Ω, which is a topological tensor product of $\mathcal{S}$ and the constant presheaf generated by the space E_k. It follows that the n-th column $\mathcal{N}^n$ of the bicomplex $\overline{\mathcal{N}}$ is $\prod\limits_{\alpha\in M_n} \mathcal{S}(V_\alpha)\underset{A}{\widehat{\otimes}}\mathcal{P}$.

4.3. THE DOMINATING COMPLEX AND SPECTRUM. Let $\mathcal{S}$ be a Fréchet sheaf over a topological space Ω. We say that Ω is an *$\mathcal{S}$-space* if Ω has a countable base (called *$\mathcal{S}$-base*) $\mathcal{B}$ comprising open $\mathcal{S}$-acyclic subsets such that $U \cap U' \in \mathcal{B}$ whenever $U, U' \in \mathcal{B}$.

EXAMPLE 4.3. Let $\Omega = \mathbb{C}^n$, X a Fréchet space, $\mathcal{S} = \mathcal{O}\widehat{\otimes}X$ and let $\mathcal{B}$ be a countable base in $\mathbb{C}^n$ of all finite intersections of polydisks centered at points in $\mathbb{C}^n (= \mathbb{R}^{2n})$ with rational coordinates, and rational radii. By Corollary 3.2, $\mathcal{B}$ consists of open $\mathcal{O}\widehat{\otimes}X$-acyclic subsets. Thereby, $\mathbb{C}^n$ is a $\mathcal{O}\widehat{\otimes}X$-space. Note also that the space $\mathbb{C}^n$ itself is a $\mathcal{O}\widehat{\otimes}X$-acyclic thanks to Corollary 3.2.

Without any doubt each $\mathcal{S}$-space is locally $\mathcal{S}$-acyclic with a countable base. Note that an open subset of a $\mathcal{S}$-space can be covered by a countable family of open $\mathcal{S}$-acyclic subsets.

Now let Ω be an $\mathcal{S}$-space (with an $\mathcal{S}$-base $\mathcal{B}$), where $\mathcal{S}$ is a Fréchet algebra sheaf over Ω and let $X \in A$-mod be a finitely-free Fréchet module over the algebra A of global sections of $\mathcal{S}$. Take an open subset $U \subseteq \Omega$ such that $\sigma(\mathcal{S}, X) \subseteq U$, and let $\mathfrak{U} = \{U_i\}_{i\in\mathbb{N}}$ be a countable cover of U by $U_i \in \mathcal{B}$. For each $\lambda \in \operatorname{res}(\mathcal{S}, X)$, let $D_\lambda \in \mathcal{B}$ be an open neighborhood of λ such that $\mathcal{S}(W) \perp_A X$ for each open $\mathcal{S}$-acyclic subset $W \subseteq D_\lambda$ (see Definition 4.2). Since $\Omega = \bigcup\limits_i U_i \cup \bigcup\limits_{\lambda\in\operatorname{res}(\mathcal{S},X)} D_\lambda$ and all $U_i, D_\lambda \in \mathcal{B}$, it follows that $\Omega = \bigcup\limits_i U_i \cup \bigcup\limits_n D_{\lambda_n}$ for a certain countable subset $\{\lambda_n\} \subseteq \operatorname{res}(\mathcal{S}, X)$. We set $D_i = D_{\lambda_i}$, $i \in \mathbb{N}$, and $\mathfrak{D} = \{D_i\}_{i\in\mathbb{N}}$. Then $\mathfrak{V} = \{V_i\}_{i\in\mathbb{N}} = \mathfrak{U} \cup \mathfrak{D}$ is a countable cover of Ω with $V_i \in \mathcal{B}$, where $V_{2i-1} = U_i$ and $V_{2i} = D_i$. Consider a bijection mapping $\varphi : \mathbb{N} \to J$, $\varphi(i) = 2i-1$, where J is the set of all positive odd integers. The mapping φ can be lifted up to a morphism of Čech complexes $\overline{\varphi} : \mathcal{C}^\bullet(\mathfrak{V}) \to \mathcal{C}^\bullet(\mathfrak{U})$ (see Subsection 4.2).

LEMMA 4.4. *If* $(\mathcal{C}^\bullet(\mathfrak{V}), \varepsilon) \gg X$ *then* $(\mathcal{C}^\bullet(\mathfrak{U}), \varepsilon) \gg X$.

Proof. The morphism $\overline{\varphi}$ generates the exact sequence of bicomplexes

$$0 \to \overline{\mathcal{N}} \to \overline{\mathcal{C}^\bullet(\mathfrak{V})\underset{A}{\widehat{\otimes}}\mathcal{P}} \to \overline{\mathcal{C}^\bullet(\mathfrak{U})\underset{A}{\widehat{\otimes}}\mathcal{P}} \to 0, \tag{4.1}$$

where $\mathcal{P} = \{A\widehat{\otimes}E_k, \delta^{-k} : 0 \leqslant k \leqslant n\}$ is a finite A-free Tor-resolution of X. Moreover, the n-th column $\mathcal{N}^n$ of the kernel $\overline{\mathcal{N}}$ is $\prod\limits_{\alpha\in M_n} \mathcal{S}(V_\alpha)\underset{A}{\widehat{\otimes}}\mathcal{P}$, where $\mathcal{E} = \{E_k\}$. If $\alpha \in M_n$ then $V_\alpha \subseteq V_{2i} = D_i$ for a certain i. All V_α are open $\mathcal{S}$-acyclic subsets, for $V_\alpha \in \mathcal{B}$. By Definition 4.2, $\mathcal{S}(V_\alpha) \perp_A X$, which in turn implies the exactness of the complex $\mathcal{S}(V_\alpha)\underset{A}{\widehat{\otimes}}\mathcal{P}$. Consequently, $\mathcal{N}^n = \prod\limits_{\alpha\in M_n} \mathcal{S}(V_\alpha)\underset{A}{\widehat{\otimes}}\mathcal{P}$ is an exact complex and the bicomplex $\overline{\mathcal{N}}$ with exact columns has its exact total complex. Thus (4.1) is an exact sequence of bicomplexes such that the total complex of $\overline{\mathcal{N}}$ is exact. Therefore, if $(\mathcal{C}^\bullet(\mathfrak{V}), \varepsilon) \gg X$ (see Definition 2.1) then $(\mathcal{C}^\bullet(\mathfrak{U}), \varepsilon) \gg X$ thanks to Lemma 2.3. ∎

PROPOSITION 4.5. *Let Ω be a $\mathcal{S}$-space, where $\mathcal{S}$ is a nuclear Fréchet algebra sheaf over Ω, $\mathcal{B}$ a $\mathcal{S}$-base in Ω, and let $\mathfrak{V} = \{V_i\}_{i\in\mathbb{N}} \subseteq \mathcal{B}$ be a countable cover of Ω. If Ω is $\mathcal{S}$-acyclic and $X \in A$-*mod *is a finitely-free Fréchet module then $\mathcal{C}^\bullet(\mathfrak{V})$ is a dominating over the module X complex, where $A = \mathcal{S}(\Omega)$.*

Proof. First, note that $(A, \mathrm{id}) \gg X$ by Lemma 2.2, and the morphism $\overline{\varepsilon} : A \to \mathcal{C}^\bullet(\mathfrak{V})$ is a morphism of the augmented complexes of the right A-modules. Take a free Tor-resolution $\mathcal{P} = \{P^{-k}, \delta^{-k}\}$ of X, where $P^{-k} = A\widehat{\otimes}E_k$ for some $E_k \in \mathbf{FS}$, $0 \leqslant k \leqslant n$, and let

$$\overline{\varepsilon}\widehat{\otimes}_A 1 : \overline{A\underset{A}{\widehat{\otimes}}\mathcal{P}} \to \overline{\mathcal{C}^\bullet(\mathfrak{V})\underset{A}{\widehat{\otimes}}\mathcal{P}}$$

be a morphism between bicomplexes. Let us note that each row $\mathcal{C}^\bullet(\mathfrak{V})\underset{A}{\widehat{\otimes}}(A\widehat{\otimes}E_k)$ of the bicomplex $\overline{\mathcal{C}^\bullet(\mathfrak{V})\underset{A}{\widehat{\otimes}}\mathcal{P}}$ is just $\mathcal{C}^\bullet(\mathfrak{V})\widehat{\otimes}E_k$. The relevant row of the first bicomplex $\overline{A\underset{A}{\widehat{\otimes}}\mathcal{P}}$ is $0 \to A\widehat{\otimes}E_k \to 0 \to 0 \to \cdots$ (the first bicomplex has only one nontrivial column $\mathcal{P}$).

By assumption, $\mathfrak{V} = \{V_i\}_{i\in\mathbb{N}} \subseteq \mathcal{B}$, thereby all finite intersections $V_\alpha = V_{i_0} \cap \cdots \cap V_{i_n}$ taken in $\mathfrak{V}$ are $\mathcal{S}$-acyclic, that is, $H^n(V_\alpha, \mathcal{S}) = \{0\}$, $n \in \mathbb{N}$. By Leray theorem ([19], Chapter 2 Theorem 5.4.1), $H^n(\mathfrak{V}, \mathcal{S}) = H^n(\Omega, \mathcal{S})$ up to an isomorphism for all $n \in \mathbb{Z}_+$. Since Ω is a $\mathcal{S}$-acyclic space, we conclude that $H^n(\mathfrak{V}, \mathcal{S}) = \{0\}$, $n \in \mathbb{N}$. Thus $0 \to A \xrightarrow{\varepsilon} \mathcal{C}^\bullet(\mathfrak{V})$ is an exact complex in **FS**. But $\mathcal{S}$ is a nuclear Fréchet algebra sheaf, therefore A and all members of the complex $\mathcal{C}^\bullet(\mathfrak{V})$ are nuclear spaces. It follows that the complex

$$0 \to A\widehat{\otimes}E_k \overset{\varepsilon\otimes 1}{\to} \mathcal{C}^\bullet(\mathfrak{V})\widehat{\otimes}E_k$$

remains exact ([25], Proposition 4.2). Thus $\overline{\varepsilon}\widehat{\otimes}_A 1$ is an embedding and all rows of the quotient bicomplex $\overline{\mathcal{M}} = \overline{\mathcal{C}^\bullet(\mathfrak{V})\underset{A}{\widehat{\otimes}}\mathcal{P}} / \operatorname{im}(\overline{\varepsilon}\widehat{\otimes}_A 1)$ are exact complexes

$$\mathcal{C}^\bullet(\mathfrak{V})\widehat{\otimes}E_k/\operatorname{im}(\overline{\varepsilon}\otimes 1) : 0 \to \mathcal{C}^0(\mathfrak{V})\widehat{\otimes}E_k/\operatorname{im}(\varepsilon\otimes 1) \to \mathcal{C}^1(\mathfrak{V})\widehat{\otimes}E_k \to \mathcal{C}^2(\mathfrak{V})\widehat{\otimes}E_k \to \cdots.$$

Thus the bicomplex $\overline{\mathcal{M}}$ with exact rows has the exact total complex. But

$$0 \to \overline{A\underset{A}{\widehat{\otimes}}\mathcal{P}} \to \overline{\mathcal{C}^\bullet(\mathfrak{V})\underset{A}{\widehat{\otimes}}\mathcal{P}} \to \overline{\mathcal{M}} \to 0$$

is an exact sequence of bicomplexes, where $(A, \mathrm{id}) \gg X$ and the total complex of $\overline{\mathcal{M}}$ is exact. Then $(\mathcal{C}^\bullet(\mathfrak{V}), \varepsilon) \gg X$ thanks to Lemma 2.3. ∎

4.4. FUNCTIONAL CALCULUS. Now we suggest a functional calculus for a finitely free Fréchet module over an algebra of the global sections of a Fréchet sheaf.

THEOREM 4.6. *Let Ω be a $\mathcal{S}$-space, where $\mathcal{S}$ is a nuclear Fréchet algebra sheaf over Ω, $X \in \mathcal{S}(\Omega)$*-mod *a finitely-free Fréchet module and let U be an open neighborhood of Putinar spectrum $\sigma(\mathcal{S}, X)$ in Ω. If Ω is $\mathcal{S}$-acyclic then $X \in \mathcal{S}(U)$*-mod *and its $\mathcal{S}(\Omega)$-module structure via pullback along the restriction homomorphism $\mathcal{S}(\Omega) \to \mathcal{S}(U)$ is reduced to the original one.*

Proof. Take an $\mathcal{S}$-base $\mathcal{B}$ in Ω and a countable cover $\mathfrak{U} \subseteq \mathcal{B}$ of U, which is extended up to a countable cover $\mathfrak{V}$ of the whole space Ω as suggested above. Note that $\mathfrak{V} \subseteq \mathcal{B}$. By Proposition 4.5, $(\mathcal{C}^\bullet(\mathfrak{V}), \varepsilon) \gg X$, which in turn implies that $(\mathcal{C}^\bullet(\mathfrak{U}), \varepsilon) \gg X$ by virtue of Lemma 4.4. But $\mathcal{C}^\bullet(\mathfrak{U}) \in \overline{\mathcal{S}(U)\text{-}\operatorname{mod}\text{-}\mathcal{S}(\Omega)}$ and the augmentation $\varepsilon : \mathcal{S}(\Omega) \to \mathcal{C}^0(\mathfrak{U})$ is a morphism of left $\mathcal{S}(\Omega)$-modules. It remains to apply Theorem 2.4. ∎

To demonstrate how the assertion from Theorem 4.6 works in a particular case, we consider a typical example, when Ω is a Stein domain in $\mathbb{C}^n$ and $\mathcal{S} = \mathcal{O}$ is the sheaf of germs of holomorphic functions over Ω. By definition, Ω is a $\mathcal{O}$-acyclic. Moreover, Ω is a $\mathcal{O}$-space and $\mathcal{O}$ is a nuclear Fréchet algebra sheaf over Ω. Let $X \in \mathcal{O}(\Omega)$-mod be a Fréchet module. Then X is a finitely-free $\mathcal{O}(\Omega)$-module by (see Subsection 1.1). Appealing to Theorem 4.6, we conclude that X turns into a left Fréchet $\mathcal{O}(U)$-module compatible with the $\mathcal{O}(\Omega)$-module structure whenever U is an open neighborhood in Ω of Putinar spectrum $\sigma(\mathcal{S}, X)$

(see Theorem 5.1.5 of [16]). Below we propose a noncommutative version of this result.

5. THE SHEAF $\mathfrak{T}_{\mathfrak{g}}$ OF GERMS OF FORMALLY-RADICAL FUNCTIONS

In this section we introduce a (noncommutative) Fréchet algebra sheaf $\mathfrak{T}_{\mathfrak{g}}$ of germs of formally radical functions in elements of a positively graded nilpotent Lie algebra $\mathfrak{g}$. The sheaf $\mathfrak{T}_{\mathfrak{g}}$ will possess all conditions of the functional calculus in Theorem 4.6. First we remind the basic properties of the formally radical functions in elements of a nilpotent Lie algebra investigated in [12], [8].

5.1. FORMALLY-RADICAL FUNCTIONS IN ELEMENTS OF A NILPOTENT LIE ALGEBRA. Everywhere below we fix a finite dimensional positively graded nilpotent Lie algebra $\mathfrak{g}$ and its basis $e = (e_1, \dots, e_n)$ which obeys to that grading. Since each nilpotent Lie algebra can be presented as a quotient of a positively graded nilpotent Lie algebra, that will motivate our choice $\mathfrak{g}$ as the noncommutative variable space. Indeed, let $\mathfrak{h}$ be a finite dimensional nilpotent Lie algebra generated by $x_1, \dots, x_m$. Consider the quotient $\mathfrak{g}$ of the free Lie algebra generated by m elements $e_1, \dots, e_m$ modulo the appropriate degree its lower central series. Evidently, $\mathfrak{g}$ admits positive grading with numbers $1, \dots, c$, where c is the nilpotence degree of $\mathfrak{h}$. Moreover, there exists a Lie epimorphism $\tau : \mathfrak{g} \to \mathfrak{h}$ such that $\tau(e_i) = x_i$, $1 \leqslant i \leqslant m$.

Let $\mathfrak{r}_e \subseteq \mathcal{U}(\mathfrak{g})$ be a subset of all radical monomials $e_{\mathrm{r}}^{J_{\mathrm{r}}} = e_{m+1}^{j_{m+1}} \cdots e_n^{j_n}$ ($1 = e_{\mathrm{r}}^{J_{\mathrm{r}}}$ for $J_{\mathrm{r}} = (0, \dots, 0)$), $J_{\mathrm{r}} \in \mathbb{Z}_+^{n-m}$. Evidently, $\mathfrak{r}_e$ is a subset of the set $\{e^J\}$ of all ordered monomials (see Section 1). If D_r is a polydisk in the character space $\Delta(\mathfrak{g}) (= \mathbb{C}^m)$ of multiradius r centered at the origin then the Fréchet algebra $\mathcal{F}_{\mathfrak{g}}(D_r)$ of all *formally radical functions in elements of* $\mathfrak{g}$ is defined as the Fréchet space $\mathcal{O}(D_r)[[e_{m+1}, \dots, e_n]]$ of all formal power series over the Fréchet space $\mathcal{O}(D_r)$ in the radical variables e_{r}. Thus each $f \in \mathcal{F}_{\mathfrak{g}}(D_r)$ has unique formal power series expansion $f = \sum_{J_{\mathrm{r}}} f_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}}$, where $f_{J_{\mathrm{r}}} \in \mathcal{O}(D_r)$. If $p = \sum_J x_J e^J \in \mathcal{U}(\mathfrak{g})$ then $p = \sum_{J_{\mathrm{r}}} p_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}}$, where $p_{J_{\mathrm{r}}} = \sum_{I_{\mathrm{s}}} x_{I_{\mathrm{s}} \cup J_{\mathrm{r}}} e^{I_{\mathrm{s}}}$. If we identify $p_{J_{\mathrm{r}}}$ with a polynomial $\sum_{I_{\mathrm{s}}} x_{I_{\mathrm{s}} \cup J_{\mathrm{r}}} z^{I_{\mathrm{s}}}$ in $\mathcal{O}(D_r)$ then $p = \sum_{J_{\mathrm{r}}} p_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}} \in \mathcal{F}_{\mathfrak{g}}(D_r)$. Thus $\mathcal{U}(\mathfrak{g})$ is a dense subspace in $\mathcal{F}_{\mathfrak{g}}(D_r)$. Moreover, the multiplication on $\mathcal{U}(\mathfrak{g})$ can uniquely be lifted to the jointly continuous (noncommutative) multiplication $*$ on $\mathcal{F}_{\mathfrak{g}}(D_r)$ called the nilpotent convolution [8]. Whence $\mathcal{F}_{\mathfrak{g}}(D_r)$ is a Fréchet algebra and $\mathcal{U}(\mathfrak{g})$ is its dense subalgebra. Moreover, the space of all continuous characters on $\mathcal{F}_{\mathfrak{g}}(D_r)$ is identified with the polydisk D_r [8]. If X is a Banach space then the space $\mathcal{F}_{\mathfrak{g}}(D_r, X)$ of *X-valued formally radical functions in elements of* $\mathfrak{g}$ is defined as the projective tensor product $\mathcal{F}_{\mathfrak{g}}(D_r) \widehat{\otimes} X$. It is not so hard to prove that the set $\{e^J\}$ of all ordered monomials in $\mathcal{U}(\mathfrak{g})$ is an absolute X-valued basis in $\mathcal{F}_{\mathfrak{g}}(D_r, X)$ (for the details see [8]), that is, each $\overline{f} \in \mathcal{F}_{\mathfrak{g}}(D_r, X)$ has unique power series expansion $\overline{f} = \sum\limits_J x_J e^J$, $x_J \in X$, as an absolutely convergent in $\mathcal{F}_{\mathfrak{g}}(D_r, X)$ power series. Without any doubt,

$$\mathcal{F}_{\mathfrak{g}}(D_r, X) = \mathcal{O}(D_r, X)[[e_{\mathrm{r}}]]$$

up to a topological isomorphism.

Now take a point $a \in \Delta(\mathfrak{g})$ and let $\mathfrak{g} - a$ be a Lie subalgebra in $\mathcal{U}(\mathfrak{g})$ comprising all elements $u - a(u)$, $u \in \mathfrak{g}$. We set $\mathcal{F}_{\mathfrak{g}}(D_{a,r}) = \mathcal{F}_{\mathfrak{g}-a}(D_r)$ for a polydisk $D_{a,r} \subseteq \Delta(\mathfrak{g})$ of multiradius r centered at a. If $D_{b,v} \subset D_{a,r} \subseteq \Delta(\mathfrak{g})$ are polydisks then we have a well defined restriction mapping

$$P^{a,r}_{b,v} : \mathcal{F}_{\mathfrak{g}}(D_{a,r}) \to \mathcal{F}_{\mathfrak{g}}(D_{b,v}), \quad P^{a,r}_{b,v}\Big(\sum_{J_{\mathrm{r}}} f_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}}\Big) = \sum_{J_{\mathrm{r}}} (f_{J_{\mathrm{r}}}|_{D_{b,v}}) e_{\mathrm{r}}^{J_{\mathrm{r}}},$$

where $f_{J_{\mathrm{r}}}|_{D_{b,v}}$ is the restriction of the holomorphic function $f_{J_{\mathrm{r}}} \in \mathcal{O}(D_{a,r})$. This is a continuous algebra homomorphism [8]. Take arbitrary points a, b, c in $\Delta(\mathfrak{g})$, $f \in \mathcal{F}_{\mathfrak{g}}(D_{a,r})$, $g \in \mathcal{F}_{\mathfrak{g}}(D_{b,v})$, and let $D_{c,q} \subset D_{a,r} \cap D_{b,v}$. Then $f = \sum\limits_{J_{\mathrm{r}}} f_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}}$, $g = \sum\limits_{J_{\mathrm{r}}} g_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}}$, where $f_{J_{\mathrm{r}}} \in \mathcal{O}(D_{a,r})$, $g_{J_{\mathrm{r}}} \in \mathcal{O}(D_{b,v})$. Assume that $P^{a,r}_{c,q}(f) = P^{b,v}_{c,q}(g)$, where $P^{a,r}_{c,q} : \mathcal{F}_{\mathfrak{g}}(D_{a,r}) \to \mathcal{F}_{\mathfrak{g}}(D_{c,q})$ and $P^{b,v}_{c,q} : \mathcal{F}_{\mathfrak{g}}(D_{b,v}) \to \mathcal{F}_{\mathfrak{g}}(D_{c,q})$ are the algebra homomorphisms. Then $f_{J_{\mathrm{r}}}|_{D_{c,q}} = g_{J_{\mathrm{r}}}|_{D_{c,q}}$ for all J_{r}. Therefore, $f_{J_{\mathrm{r}}}|_{D_{a,r} \cap D_{b,v}} = g_{J_{\mathrm{r}}}|_{D_{a,r} \cap D_{b,v}}$. In this situation, we write $f|_{D_{a,r} \cap D_{b,v}} = g|_{D_{a,r} \cap D_{b,v}}$. Let U be a nonempty open subset in $\mathbb{C}^n$. Then U has a countable cover $U = \bigcup\limits_i D_i$ by open polydisks $D_i = D_{a_i, r_i}$. Let $\mathcal{F}_{\mathfrak{g}}(U)$ be a subspace of the topological direct product $\prod\limits_i \mathcal{F}_{\mathfrak{g}}(D_i)$ comprising all compatible elements $\{f_i\}_i$, that is, $f_i|_{D_i \cap D_j} = f_j|_{D_i \cap D_j}$ for all i, j. Since the "restriction" mappings P^a_b are continuous, it follows that $\mathcal{F}_{\mathfrak{g}}(U)$ is a closed subspace, thereby, $\mathcal{F}_{\mathfrak{g}}(U)$ is a Fréchet space. The nilpotent convolution is extended up to $\mathcal{F}_{\mathfrak{g}}(U)$ in the canonical way:

$$\{f_i\}_i * \{g_i\}_i = \{f_i * g_i\}_i.$$

Then $(f_i * g_i)|_{D_i \cap D_j} = f_i|_{D_i \cap D_j} * g_i|_{D_i \cap D_j} = f_j|_{D_i \cap D_j} * g_j|_{D_i \cap D_j} = (f_j * g_j)|_{D_i \cap D_j}$ for all i,j. Thus $\mathcal{F}_{\mathfrak{g}}(U)$ is a Fréchet algebra with respect to the nilpotent convolution. Moreover,

$$\mathcal{F}_{\mathfrak{g}}(U) = \mathcal{O}(U)[[e_{\mathrm{r}}]] \tag{5.1}$$

as a Fréchet space [8]. In particular, the space $\mathcal{F}_{\mathfrak{g}}(U)$ does not depend upon the particular choice of a polydisk cover $\{D_i\}$ of U. Therefore, if $U = D_{a,r}$ is a polydisk then $\mathcal{F}_{\mathfrak{g}}(U) = \mathcal{F}_{\mathfrak{g}}(D_{a,r})$. If X is a Fréchet space then

$$\mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} X = \mathcal{O}(U, X) \widehat{\otimes} \mathbb{C}[[e_{\mathrm{r}}]] = \mathcal{O}(U, X)[[e_{\mathrm{r}}]] \tag{5.2}$$

up to a topological isomorphism of the Fréchet spaces. In particular, each $\overline{f} \in \mathcal{F}_{\mathfrak{g}}(U)\widehat{\otimes}X$ has unique expansion $\overline{f} = \sum_{J_{\mathrm{r}}} \overline{f}_{J_{\mathrm{r}}} e_{\mathrm{r}}^{J_{\mathrm{r}}}$ as a formal power series, where $\overline{f}_{J_{\mathrm{r}}} \in \mathcal{O}(U, X)$.

Finally, if $\cdots \longleftarrow X_{n-1} \overset{T_{n-1}}{\longleftarrow} X_n \overset{T_n}{\longleftarrow} X_{n+1} \longleftarrow \cdots$ is an exact sequence of Fréchet spaces then the sequence

$$\cdots \longleftarrow \mathcal{F}_{\mathfrak{g}}(U)\widehat{\otimes}X_{n-1} \overset{1\otimes T_{n-1}}{\longleftarrow} \mathcal{F}_{\mathfrak{g}}(U)\widehat{\otimes}X_n \overset{1\otimes T_n}{\longleftarrow} \mathcal{F}_{\mathfrak{g}}(U)\widehat{\otimes}X_{n+1} \longleftarrow \cdots \tag{5.3}$$

remains exact. Indeed, it is well known ([20], 2.4.16) that the sequence

$$\cdots \longleftarrow \mathcal{O}(U, X_{n-1}) \overset{1\otimes T_{n-1}}{\longleftarrow} \mathcal{O}(U, X_n) \overset{1\otimes T_n}{\longleftarrow} \mathcal{O}(U, X_{n+1}) \longleftarrow \cdots$$

remains exact. It remains to note that the functor $\circ\widehat{\otimes}\mathbb{C}[[e_{\mathrm{r}}]]$ applied to the latter complex does not change its exactness [25], for $\mathbb{C}[[e_{\mathrm{r}}]]$ is a Fréchet nuclear space.

5.2. THE KOSZUL RESOLUTIONS. Let U be a domain in $\Delta(\mathfrak{g})$ and let $\alpha : \mathfrak{g} \to \mathcal{L}(X)$ be a Lie representation of $\mathfrak{g}$ in a Fréchet space X. Then $\mathcal{F}_{\mathfrak{g}}(U)\widehat{\otimes}X$ turns out to be a $\mathfrak{g}$-module via the representation

$$\rho_{U,X} : \mathfrak{g} \to \mathcal{L}(\mathcal{F}_{\mathfrak{g}}(U)\widehat{\otimes}X), \quad \rho_{U,X}(u)(f \otimes x) = f \otimes \alpha(u)x - f * u \otimes x,$$

$f \in \mathcal{F}_{\mathfrak{g}}(U), x \in X, u \in \mathfrak{g}$, that is, $\rho_{U,X}(u) = 1 \otimes L_{\alpha(u)} - R_u \otimes 1$, $u \in \mathfrak{g}$. Hence we have a Koszul complex $\mathrm{Kos}(\mathcal{F}_{\mathfrak{g}}(U)\widehat{\otimes}X, \rho_{U,X})$ (see Subsection 1.2) associated by the representation $\rho_{U,X}$. Put $\rho_U = \rho_{U,\mathcal{F}_{\mathfrak{g}}(U)}$. The Koszul complex $\mathrm{Kos}(\mathcal{F}_{\mathfrak{g}}(U)\widehat{\otimes} \mathcal{F}_{\mathfrak{g}}(U), \rho_U)$ is augmented by means of the multiplication mapping

$$\pi_U : \mathcal{F}_{\mathfrak{g}}(U)\widehat{\otimes}\mathcal{F}_{\mathfrak{g}}(U) \to \mathcal{F}_{\mathfrak{g}}(U), \quad f \otimes g \mapsto fg.$$

It was proved in [8], [12] that the chain complex

$$0 \leftarrow \mathcal{F}_{\mathfrak{g}}(D) \overset{\pi_D}{\longleftarrow} \mathrm{Kos}(\mathcal{F}_{\mathfrak{g}}(D)\widehat{\otimes}\mathcal{F}_{\mathfrak{g}}(D), \rho_D) \tag{5.4}$$

is admissible whenever D is a polydisk in $\Delta(\mathfrak{g})$, so, $\mathrm{Kos}(\mathcal{F}_{\mathfrak{g}}(D)\widehat{\otimes}\mathcal{F}_{\mathfrak{g}}(D), \rho_D)$ is a free $\mathcal{F}_{\mathfrak{g}}(D)$-bimodule resolution of the Fréchet algebra $\mathcal{F}_{\mathfrak{g}}(D)$. Moreover, $\mathcal{F}_{\mathfrak{g}}(D) \perp_{\mathcal{F}_{\mathfrak{g}}} X$ implies that $D \cap \sigma(\mathfrak{g}, X) = \varnothing$, and, whenever X is a Banach space (see [5])

$$\mathcal{F}_{\mathfrak{g}}(D) \perp_{\mathcal{F}_{\mathfrak{g}}} X \quad \Longleftrightarrow \quad D \cap \sigma(\mathfrak{g}, X) = \varnothing. \tag{5.5}$$

LEMMA 5.1. *Let $X \in \mathcal{F}_{\mathfrak{g}}$-mod be a Fréchet module, Y a Fréchet space, and let D be a polydisk in $\Delta(\mathfrak{g})$. Then $(Y\widehat{\otimes}\mathcal{F}_{\mathfrak{g}}(D)) \perp_{\mathcal{F}_{\mathfrak{g}}} X$ if and only if the complex $Y\widehat{\otimes}\,\mathrm{Kos}(\mathcal{F}_{\mathfrak{g}}(D)\widehat{\otimes}X, \rho_{D,X})$ is exact, where $Y\widehat{\otimes}\mathcal{F}_{\mathfrak{g}}(D)$ is considered to be the right $\mathcal{F}_{\mathfrak{g}}$-module.*

Proof. Since all members of the complex (5.4) are free right $\mathcal{F}_{\mathfrak{g}}(D)$-modules, it follows that the complex (5.4) splits as a complex in mod-$\mathcal{F}_{\mathfrak{g}}(D)$. In particular, so is the complex $0 \leftarrow \mathcal{F}_{\mathfrak{g}} \xleftarrow{\pi_{\mathbb{C}^m}} \operatorname{Kos}(\mathcal{F}_{\mathfrak{g}} \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}, \rho_{\mathbb{C}^m})$ in mod-$\mathcal{F}_{\mathfrak{g}}$ (just put $D = \mathbb{C}^m$). Applying the functor $\circ \underset{\mathcal{F}_{\mathfrak{g}}}{\widehat{\otimes}} X$ to the latter complex, we obtain that the complex

$$0 \leftarrow X \xleftarrow{\pi_X} \operatorname{Kos}(\mathcal{F}_{\mathfrak{g}} \widehat{\otimes} X, \rho_{\mathbb{C}^m, X}) \tag{5.6}$$

is admissible, that is, X has the free Koszul resolution in $\mathcal{F}_{\mathfrak{g}}$-mod. With $\mathcal{F}_{\mathfrak{g}} \subseteq \mathcal{F}_{\mathfrak{g}}(D)$ in mind, infer $\mathcal{F}_{\mathfrak{g}}(D) \in \mathcal{F}_{\mathfrak{g}}$-mod-$\mathcal{F}_{\mathfrak{g}}$. Based upon the resolution (5.6), we infer that $(Y \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}(D)) \perp_{\mathcal{F}_{\mathfrak{g}}} X$ if and only if the complex $(Y \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}(D)) \underset{\mathcal{F}_{\mathfrak{g}}}{\widehat{\otimes}} \operatorname{Kos}(\mathcal{F}_{\mathfrak{g}} \widehat{\otimes} X, \rho_{\mathbb{C}^m, X})$ is exact. Evidently,

$$(Y \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}(D)) \underset{\mathcal{F}_{\mathfrak{g}}}{\widehat{\otimes}} \operatorname{Kos}(\mathcal{F}_{\mathfrak{g}} \widehat{\otimes} X, \rho_{\mathbb{C}^m, X}) = Y \widehat{\otimes} \operatorname{Kos}(\mathcal{F}_{\mathfrak{g}}(D) \widehat{\otimes} X, \rho_{D,X})$$

up to a topological isomorphism. The rest is clear. ∎

PROPOSITION 5.2. *Let D be a polydisk in $\Delta(\mathfrak{g})$, U an open subset in $\Delta(\mathfrak{g})$ such that $U \subseteq D$, and let $X \in \mathcal{F}_{\mathfrak{g}}$-mod be a Fréchet module. If $\mathcal{F}_{\mathfrak{g}}(D) \perp_{\mathcal{F}_{\mathfrak{g}}} X$ then $\mathcal{F}_{\mathfrak{g}}(U) \perp_{\mathcal{F}_{\mathfrak{g}}} X$.*

Proof. Assume that $\mathcal{F}_{\mathfrak{g}}(D) \perp_{\mathcal{F}_{\mathfrak{g}}} X$. First, let us prove that we have the transversality $(\mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}(D)) \perp_{\mathcal{F}_{\mathfrak{g}}} X$. By Lemma 5.1, one should prove that the complex $\mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} \operatorname{Kos}(\mathcal{F}_{\mathfrak{g}}(D) \widehat{\otimes} X, \rho_{D,X})$ is exact. But the Koszul complex $\operatorname{Kos}(\mathcal{F}_{\mathfrak{g}}(D) \widehat{\otimes} X, \rho_{D,X})$ is exact by virtue of Lemma 5.1 (just put $Y = \mathbb{C}$). It follows that so is the complex $\mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} \operatorname{Kos}(\mathcal{F}_{\mathfrak{g}}(D) \widehat{\otimes} X, \rho_{D,X})$ thanks to (5.3). Thus $\mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}(D) \perp_{\mathcal{F}_{\mathfrak{g}}} X$. Being $\circ \underset{\mathcal{F}_{\mathfrak{g}}}{\widehat{\otimes}} X$ an additive functor, we derive that $\mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}(D) \otimes \bigwedge^k \mathfrak{g} \perp_{\mathcal{F}_{\mathfrak{g}}} X$ for all $k \geqslant 0$.

Undoubtedly, $\mathcal{F}_{\mathfrak{g}}(U) \in$ mod-$\mathcal{F}_{\mathfrak{g}}(D)$ (via the restriction homomorphism $\mathcal{F}_{\mathfrak{g}}(D) \to \mathcal{F}_{\mathfrak{g}}(U)$). Since all members of the complex (5.4) are free left $\mathcal{F}_{\mathfrak{g}}(D)$-modules, it follows that the complex (5.4) splits as a complex in $\mathcal{F}_{\mathfrak{g}}(D)$-mod. Applying the functor $\mathcal{F}_{\mathfrak{g}}(U) \underset{\mathcal{F}_{\mathfrak{g}}(D)}{\widehat{\otimes}} \circ$ to the complex (5.4), we obtain that the complex

$$0 \leftarrow \mathcal{F}_{\mathfrak{g}}(U) \longleftarrow \mathcal{F}_{\mathfrak{g}}(U) \underset{\mathcal{F}_{\mathfrak{g}}(D)}{\widehat{\otimes}} \operatorname{Kos}(\mathcal{F}_{\mathfrak{g}}(D) \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}(D), \rho_D)$$

is admissible. But

$$\mathcal{F}_{\mathfrak{g}}(U) \underset{\mathcal{F}_{\mathfrak{g}}(D)}{\widehat{\otimes}} \operatorname{Kos}(\mathcal{F}_{\mathfrak{g}}(D) \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}(D), \rho_D) = \operatorname{Kos}(\mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}(D), \rho_{\mathcal{F}_{\mathfrak{g}}(U), D})$$

up to a topological isomorphism. Consequently, we have an admissible complex

$$0 \leftarrow \mathcal{F}_{\mathfrak{g}}(U) \longleftarrow \operatorname{Kos}(\mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}(D), \rho_{\mathcal{F}_{\mathfrak{g}}(U), D})$$

and the left $\mathcal{F}_{\mathfrak{g}}$-module X is in the transversality relation with all members of the complex $\operatorname{Kos}(\mathcal{F}_{\mathfrak{g}}(U) \widehat{\otimes} \mathcal{F}_{\mathfrak{g}}(D), \rho_{\mathcal{F}_{\mathfrak{g}}(U), D})$. Using 3.3.8 of [20], we deduce that $\mathcal{F}_{\mathfrak{g}}(U) \perp_{\mathcal{F}_{\mathfrak{g}}} X$. ∎

5.3. THE SHEAF $\mathfrak{T}_{\mathfrak{g}}$. Consider the constant sheaf $\mathbb{C}[[\omega_1,\dots,\omega_{n-m}]]$ over $\Delta(\mathfrak{g})$ generated by the space of all formal power series in $n-m$ variables $\omega_1,\dots,\omega_{n-m}$. The topological tensor product $\mathcal{O}\widehat{\otimes}\mathbb{C}[[\omega_1,\dots,\omega_{n-m}]]$ of these sheaves is called a *sheaf of germs of formally-radical functions over* $\Delta(\mathfrak{g})$ and it is denoted by $\mathfrak{T}_{\mathfrak{g}}$. For each open subset $U\subseteq\Delta(\mathfrak{g})$, the Fréchet space $\mathfrak{T}_{\mathfrak{g}}(U)$ of all sections over U is reduced to the space $\mathcal{F}_{\mathfrak{g}}(U)=\mathcal{O}(U)[[e_{\mathrm{r}}]]$ (5.1). Thus $\mathfrak{T}_{\mathfrak{g}}$ is a nuclear Fréchet sheaf. Moreover, $\mathfrak{T}_{\mathfrak{g}}$ can be considered as a (noncommutative) Fréchet algebra sheaf with respect to the nilpotent convolution (see Subsection 5.1). Namely, $\mathfrak{T}_{\mathfrak{g}}(U)=\mathcal{F}_{\mathfrak{g}}(U)$ is the algebra of all formally-radical functions on U in elements of the Lie algebra $\mathfrak{g}$, and the restriction mapping of the sheaf $\mathfrak{T}_{\mathfrak{g}}$ is an algebra homomorphism. Thereby, $\mathfrak{T}_{\mathfrak{g}}$ is a nuclear Fréchet algebra sheaf over the character space $\Delta(\mathfrak{g})$.

We have already noted above (see Example 4.1) that the character space $\Delta(\mathfrak{g})$ is a locally $\mathfrak{T}_{\mathfrak{g}}$-acyclic space. Moreover, $\Delta(\mathfrak{g})$ is turning into a $\mathfrak{T}_{\mathfrak{g}}$-space (see Example 4.3) and $\Delta(\mathfrak{g})$ is $\mathfrak{T}_{\mathfrak{g}}$-acyclic itself, whence one may apply the functional calculus theorem stated in Theorem 4.6 to the sheaf $\mathfrak{T}_{\mathfrak{g}}$. Before that we compare the Putinar spectrum with respect to the sheaf $\mathfrak{T}_{\mathfrak{g}}$ with the Taylor spectrum of a $\mathfrak{g}$-module.

PROPOSITION 5.3. *Let $\mathcal{F}_{\mathfrak{g}}=\mathfrak{T}_{\mathfrak{g}}(\Delta(\mathfrak{g}))$ and let $X\in\mathcal{F}_{\mathfrak{g}}$-*mod *be a Fréchet module. The resolvent set* $\operatorname{res}(\mathfrak{T}_{\mathfrak{g}},X)$ *coincides with the set of those $\lambda\in\Delta(\mathfrak{g})$ such that $\mathfrak{T}_{\mathfrak{g}}(D_\lambda)\perp_{\mathcal{F}_{\mathfrak{g}}}X$ for a certain polydisk $D_\lambda\subseteq\Delta(\mathfrak{g})$ centered at λ. In particular, the Taylor spectrum of the $\mathfrak{g}$-module X is contained in the Putinar spectrum of the $\mathcal{F}_{\mathfrak{g}}$-module X with respect to the sheaf $\mathfrak{T}_{\mathfrak{g}}$, that is,*

$$\sigma(\mathfrak{g},X)\subseteq\sigma(\mathfrak{T}_{\mathfrak{g}},X)$$

and they coincide whenever X is a Banach space. Moreover, the spectrum $\sigma(\mathfrak{T}_{\mathfrak{g}},X)$ is the smallest closed set with the property $\mathfrak{T}_{\mathfrak{g}}(D)\perp_{\mathcal{F}_{\mathfrak{g}}}X$ for each open polydisk $D\subseteq\Delta(\mathfrak{g})\backslash\sigma(\mathfrak{T}_{\mathfrak{g}},X)$.

Proof. Take $\lambda\in\Delta(\mathfrak{g})$ such that $\mathcal{F}_{\mathfrak{g}}(D_\lambda)\perp_{\mathcal{F}_{\mathfrak{g}}}X$. Using Proposition 5.2, we obtain that $\mathcal{F}_{\mathfrak{g}}(U)\perp_{\mathcal{F}_{\mathfrak{g}}}X$ whenever $U\subseteq D_\lambda$ is open. Then $\lambda\in\operatorname{res}(\mathfrak{T}_{\mathfrak{g}},X)$ by Definition 4.2. Conversely, if $\lambda\in\operatorname{res}(\mathfrak{T}_{\mathfrak{g}},X)$ then $\mathfrak{T}_{\mathfrak{g}}(D_\lambda)\perp_{\mathcal{F}_{\mathfrak{g}}}X$ for a certain small polydisk D_λ centered at λ. It follows that $D_\lambda\cap\sigma(\mathfrak{g},X)=\varnothing$ [8], in particular, $\lambda\notin\sigma(\mathfrak{g},X)$. Thus $\operatorname{res}(\mathfrak{T}_{\mathfrak{g}},X)\subseteq\Delta(\mathfrak{g})\backslash\sigma(\mathfrak{g},X)$. Consequently, $\sigma(\mathfrak{g},X)\subseteq\sigma(\mathfrak{T}_{\mathfrak{g}},X)$. In order to prove the equality between spectra for the Banach space X, one should use (5.5).

It remains to prove that if $D\subseteq\operatorname{res}(\mathfrak{T}_{\mathfrak{g}},X)$ is a polydisk then $\mathfrak{T}_{\mathfrak{g}}(D)\perp_{\mathcal{F}_{\mathfrak{g}}}X$. By its very definition, for each $\lambda\in D$ we have $\mathfrak{T}_{\mathfrak{g}}(D_\lambda)\perp_{\mathcal{F}_{\mathfrak{g}}}X$ for a certain small open neighborhood $D_\lambda\subseteq D$ of λ. The latter means that we have a finite exact sequence $0\leftarrow\operatorname{Kos}(\mathfrak{T}_{\mathfrak{g}}\widehat{\otimes}X,\rho_{D,X})$ of acyclic sheaves on D (see Corollary 3.2). It follows that the complex $0\leftarrow\operatorname{Kos}(\mathfrak{T}_{\mathfrak{g}}(D)\widehat{\otimes}X,\rho_{D,X})$ is exact ([2], 2.4.3) that is, $\mathfrak{T}_{\mathfrak{g}}(D)\perp_{\mathcal{F}_{\mathfrak{g}}}X$. ∎

Now we formulate the functional calculus theorem with respect to the sheaf $\mathfrak{T}_{\mathfrak{g}}$ about the Putinar spectrum.

THEOREM 5.4. *Let $X \in \mathcal{F}_{\mathfrak{g}}$-mod be a Fréchet module and let U be an open neighborhood of the Putinar spectrum $\sigma(\mathfrak{T}_{\mathfrak{g}}, X)$ of the $\mathcal{F}_{\mathfrak{g}}$-module X with respect to the sheaf $\mathfrak{T}_{\mathfrak{g}}$. Then $X \in \mathfrak{T}_{\mathfrak{g}}(U)$-mod compatible with the $\mathcal{F}_{\mathfrak{g}}$-module structure.*

Proof. As we have noted above $\Delta(\mathfrak{g})$ is a $\mathfrak{T}_{\mathfrak{g}}$-space and it is $\mathfrak{T}_{\mathfrak{g}}$-acyclic. Moreover, X is a finitely-free Fréchet $\mathcal{F}_{\mathfrak{g}}$-module (see (5.6) in the proof of Lemma 5.1). By Theorem 4.6, the result follows. ∎

6. THE MAIN RESULT

Now we prove the main result on Taylor's functional calculus for an operator family generating a supernilpotent Lie algebra.

Let X be a Banach space and let $T = (T_1, \dots, T_m)$ be a family of bounded linear operators on X generating a finite-dimensional nilpotent Lie subalgebra $\mathfrak{g}_T$ in $\mathcal{L}(X)$. If $\mathfrak{g}_T - \lambda$ is a Lie subalgebra in $\mathcal{L}(X)$ generated by the operator family $T - \lambda = (T_1 - \lambda_1, \dots, T_m - \lambda_m)$, $\lambda_i = \lambda(T_i)$, $1 \leqslant i \leqslant m$, $\lambda \in \Delta(\mathfrak{g}_T)$, then the *Taylor spectrum $\sigma(T)$ of the operator family T* is defined (see [17]) as a set of those λ for which the Koszul complex of the $\mathfrak{g}_T - \lambda$-module X fails to be exact. As we have shown in Subsection 5.1 the Lie algebra $\mathfrak{g}_T$ is an epimorphic image of a positively graded nilpotent Lie algebra $\mathfrak{g}$ generated by m-elements $e_1, \dots, e_m$, that is, there exists a Lie epimorphism $\tau : \mathfrak{g} \to \mathfrak{g}_T$ such that $\tau(e_i) = T_i$, $1 \leqslant i \leqslant m$. We refer to $\mathfrak{g}$ as a *noncommutative variable algebra.* Thus X turns out to be a Banach module over the noncommutative variable algebra $\mathfrak{g}$. Take a triangular basis e in $\mathfrak{g}$ generated by $e_1, \dots, e_m$ and consider the Fréchet algebra $\mathcal{F}_{\mathfrak{g}} = \mathcal{F}_e(\Delta(\mathfrak{g}))$ of all formally-radical entire functions in elements of $\mathfrak{g}$. The Banach space X turns into a left Banach $\mathcal{F}_{\mathfrak{g}}$-module if and only if $\mathfrak{g}_T$ is a supernilpotent Lie subalgebra in $\mathcal{L}(X)$ [8]. Moreover, $\sigma(T) = \sigma(\mathfrak{g}, X)$ [17].

THEOREM 6.1. *Let X be a complex Banach space, $T = (T_1, \dots, T_m)$ an operator family in $\mathcal{L}(X)$ generating a supernilpotent Lie algebra, $\mathfrak{g}$ the relevant noncommutative variable algebra with its generators $e_1, \dots, e_m$, and let $U \subseteq \mathbb{C}^m$ be an open neighborhood of Taylor spectrum $\sigma(T)$ of the operator family T. Then there exists a unital continuous algebra homomorphism*

$$\alpha : \mathfrak{T}_{\mathfrak{g}}(U) \to \mathcal{L}(X)$$

from the Fréchet algebra $\mathfrak{T}_{\mathfrak{g}}(U)$ of all formally-radical functions on U in elements of $\mathfrak{g}$ into the Banach algebra $\mathcal{L}(X)$ such that $\alpha(e_i) = T_i$ for all i.

Proof. As we have just noted above X automatically turns into a left Banach $\mathcal{F}_{\mathfrak{g}}$-module and $\sigma(T) = \sigma(\mathfrak{g}, X)$. It remains to use Proposition 5.3 and Theorem 5.4. ∎

In particular, if T is a mutually commuting operator family then T generates a commutative Lie algebra F, and the relevant variable algebra is $\mathbb{C}^m$ with the canonical representation $\mathbb{C}^m \to \mathcal{L}(X)$, $z_i \mapsto T_i$, $1 \leqslant i \leqslant m$, and Theorem 6.1 is reduced to Taylor's functional calculus [26].

One of the important properties of the functional calculus is the relevant spectral mapping theorem. A general approach to this problem was investigated in [9], [13] for the functional calculi with respect to a Banach space representations of a nilpotent Lie algebra. In particular, it is proved the spectral mapping theorem with respect to the sheaf of germs of formally-radical functions whenever the considered neighborhood of Taylor spectrum (see Theorem 6.1) is a Stein $\mathfrak{T}_{\mathfrak{g}}$-rational domain in $\Delta(\mathfrak{g})$.

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