METU - NCC

	Basic Linear Algebra Midterm 2	
Code : *Math 260* Acad.Year: *2011-2012* Semester : *Spring* Date : *24.04.2012* Time : *17:40* Duration : *120 minutes*	Last Name: Name : KEY Department: Signature:	Student No.: Section:
	9 QUESTIONS ON 8 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6	7	8	9		

1. *(10 pts)* Find the matrix product

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}}_{C}$$

$$ABC = \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -6 & -6 \\ 9 & 9 \end{bmatrix}$$

2. *(12 pts)*Find all 2×2 matrices X commuting with the fixed matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (that is, $XA = AX$). Put $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$XA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$$

$$AX = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}.$$

Thus $XA = AX$ iff $c = 0$, $a = d$: $X = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

3. *(10 pts)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(x, y, z) = (y - z, x + y, z - 2x)$. Find the matrix of T relative to the standard basis e (that is, $M_e^e(T)$ or $M(T)_e^e$).

$$M_e^e(T) = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

4. *(12 pts)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(x, y, z) = (y, z, 0)$. Find bases for $Ker(T)$ and $Im(T)$ respectively.

Note that $T(x,y,z) = 0$ iff $y = z = 0$.

Therefore $\ker(T) = \{(\lambda, 0, 0) : \lambda \in \mathbb{R}\}$

$= \mathbb{R}(1,0,0) = \mathrm{Span}\{(1,0,0)\}$.

Thus $\{(1,0,0)\}$ is a basis for $\ker(T)$.

Further,

$\mathrm{im}(T) = \{(a,b,0) : a,b \in \mathbb{R}\} =$

$= \mathrm{Span}\{(1,0,0), (0,1,0)\}$, that is,

$\{(1,0,0), (0,1,0)\}$ is a basis for $\mathrm{im}(T)$.

5. *(10 pts)* Show that $T: Fun_{\mathbb{R}}(S) \to \mathbb{R}^3$, $T(f) = (f(a)+f(b), f(b)+f(c), f(c)+f(a))$ is an isomorphism of vector spaces, where $S = \{a, b, c\}$.

If $f \in \ker(T)$ then $\begin{cases} f(a)+f(b)=0 \\ f(b)+f(c)=0 \\ f(c)+f(a)=0 \end{cases}$,

which in turn implies that

$f(a) = f(b) = f(c) = 0$ or $f = 0$. Thus $\ker(T) = \{0\}$.

But $\dim(Fun_{\mathbb{R}}(S)) = |S| = 3 = \dim(\mathbb{R}^3)$.

Based on the Dimension Formula, we derive that $\dim(\operatorname{im}(T)) = 3$.

Whence $\operatorname{im}(T) = \mathbb{R}^3$.

6. *(14 pts)* Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^2$, $T(p(x)) = (p'(1), \int_0^2 3p(x)dx)$ be a linear transformation.

(a) Find the matrix ($M_e^f(T)$ or $M(T)_f^e$) of T relative to the pair (e, f) of bases, where $e = \{1, x, x^2\}$, $f = \{(1,0),(0,1)\}$.

$$M_e^f(T) = \begin{bmatrix} 0 & 1 & 2 \\ 6 & 6 & 8 \end{bmatrix}$$

(b) Using the Change of Basis Theorem, find the matrix ($M_{e'}^{f'}(T)$ or $M(T)_{f'}^{e'}$) of T relative to the new pair (e', f') of bases, where $e' = \{1+x, 1-x, 2x^2\}$, $f' = \{(1,1),(1,0)\}$.

We have $M_{e'}^{f'}(T) = M_f^{f'}(I)\, M_e^f(T)\, M_{e'}^e(I)$

with

$$M_f^{f'}(I) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad M_{e'}^e(I) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Then
$$M_{e'}^{f'}(T) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 6 & 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 & 8 \\ -6 & -5 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 0 & 16 \\ -11 & -1 & -12 \end{bmatrix}$$

7. *(12 pts)* Consider the plane $W = \{(x, y, z) \in \mathbb{R}^3 | x + y = 0\}$ with its basis vectors $f_1 = (1, -1, 0)$, $f_2 = (0, 0, 1)$.

(a) Find the matrix ($M_e^e(P)$ or $M(P)_e^e$) of the orthogonal projection $P : \mathbb{R}^3 \to \mathbb{R}^3$ onto W, relative to the standard basis e used twice. Note that

$P(\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3) = \lambda_1 f_1 + \lambda_2 f_2$ with $f_3 = (1, 1, 0)$

But $e_1 = \frac{1}{2}(f_1 + f_3)$, $e_2 = \frac{1}{2}(-f_1 + f_3)$, $e_3 = f_2 \Rightarrow$

$\Rightarrow P(e_1) = \frac{1}{2} f_1 = (\frac{1}{2}, -\frac{1}{2}, 0)$, $P(e_2) = (-\frac{1}{2}, \frac{1}{2}, 0)$,

$P(e_3) = f_2 = (0, 0, 1)$. There by

$$M_e^e(P) = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Find the matrix ($M_e^e(Q)$ or $M(Q)_e^e$) of the skew projection $Q : \mathbb{R}^3 \to \mathbb{R}^3$ onto W parallel to the vector $f_3 = (1, 0, 0)$ relative to the standard basis e used twice.

In this case we have $Q(\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3) = \lambda_1 f_1 + \lambda_2 f_2$ with $f_3 = (1, 0, 0)$. But

$e_1 = f_3$, $e_2 = -f_1 + f_3$, $e_3 = f_2 \Rightarrow Q(e_1) = (0, 0, 0)$,

$Q(e_2) = -f_1 = (-1, 1, 0)$, $Q(e_3) = f_2 = (0, 0, 1)$.

Hence

$$M_e^e(Q) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. *(10 pts)* Show that $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = (0, 2x, x + y)$ is a nilpotent transformation of index 3. Find a vector $v \in \mathbb{R}^3$ such that $f = (v, T(v), T^2(v))$ would be a basis for $\mathbb{R}^3$, and the matrix ($M_f^f(T)$ or $M(T)_f^f$) of T relative to the basis f.

Note that

$$T^2(x, y, z) = T(0, 2x, x+y) = (0, 0, 2x)$$

and $T^3(x, y, z) = T(0, 0, 2x) = \vec{0}$.

If $v = (1, 0, 0)$ then $T^2(v) = (0, 0, 2) \neq \vec{0}$.

Then $f = (v, T(v), T^2(v))$ is a basis for $\mathbb{R}^3$. Thus

$f = ((1, 0, 0), (0, 2, 1), (0, 0, 2))$ and

$$M_f^f(T) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

9. *(10 pts)* Show that $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = (y, z, x)$ is a cyclic transformation. Find a cyclic vector v for T and the matrix ($M_f^f(T)$ or $M(T)_f^f$) of T relative to the basis $f = (v, T(v), T^2(v))$.

Put $v = (1,0,0)$. Then $T(v) = (0,0,1)$ and

$T^2(v) = (0,1,0) \neq \vec{0}$. Hence

$f = ((1,0,0), (0,0,1), (0,1,0))$ is a basis

for $\mathbb{R}^3$ and

$$M_f^f(T) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$T(f_3) = T(0,1,0) = (1,0,0)$