M E T U

Northern Cyprus Campus

Math 219 **Differential Equations** Midterm **Exam I** **20.03.2014**		
Last Name: KEY Name : Student No:	Dept./Sec. : Time : 17:40 Duration : 80 *minutes*	Signature
3 QUESTIONS		TOTAL 100 POINTS

1	2	3	4	5			

Q1 (25 pts.) Determine the interval I in which the solution to IVP $\begin{cases} y' = \dfrac{ty^3}{\sqrt{1+t^2}}, \\ y(0) = 1 \end{cases}$ is defined. Hint: *Solve IVP and find its domain by performing certain estimations.*

It is a separable differentiable equation:

$$\frac{dy}{y^3} = \frac{t\,dt}{\sqrt{1+t^2}} \quad (y \neq 0) \Rightarrow -\frac{1}{2y^2} = \sqrt{1+t^2} + C$$

Since $y(0)=1$, we derive that $C = -1 - \frac{1}{2} = -\frac{3}{2}$.

Hence $\frac{1}{y^2} = 3 - 2\sqrt{1+t^2} \Rightarrow y = \dfrac{1}{\sqrt{3-2\sqrt{1+t^2}}}$

(y must be positive).

Note also that $y=0$ is not a solution to IVP.

Finally, $I = \{3 - 2\sqrt{1+t^2} \gneq 0\}$. But

$2\sqrt{1+t^2} \leq 3 \iff 1+t^2 \lneq \frac{9}{4} \iff t^2 \lneq \frac{5}{4}$, that is,

$$I = \left\{|t| \lneq \frac{\sqrt{5}}{2}\right\} = \left(-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right).$$

Q2 (25 pts.) Consider the differential equation $\left(\frac{4t^3}{y^2}+\frac{3}{y}\right)dt+\left(\frac{3t}{y^2}+4y\right)dy=0.$

15+10

(a) Find an integrating factor μ as a function of t or y. Show all details. Put $\mu=\mu(y)$.

Then $M=\mu\left(\frac{4t^3}{y^2}+\frac{3}{y}\right)$, $N=\mu\left(\frac{3t}{y^2}+4y\right)$, and

$M_y=\mu'\left(\frac{4t^3}{y^2}+\frac{3}{y}\right)-\mu\left(\frac{8t^3}{y^3}+\frac{3}{y^2}\right)=\frac{3}{y^2}\mu=N_t$. We have

$$\mu'\left(\frac{4t^3}{y^2}+\frac{3}{y}\right)=\left(\frac{6}{y^2}+\frac{8t^3}{y^3}\right)\mu\Rightarrow\mu'(4t^3y+3y^2)=$$

$$=(6y+8t^3)\mu=2(4t^3+3y)\mu\Rightarrow$$

$\mu'\cdot y\cdot(4t^3+3y)=2\mu(4t^3+3y)$. But $y=-\frac{4t^3}{3}$ is not a solution to the dif. equation. Therefore

$\frac{d\mu}{dy}\cdot y=2\mu\Rightarrow\frac{d\mu}{\mu}=\frac{2dy}{y}$ $(\mu\neq0)\Rightarrow\mu=y^2$ is an integrating factor. Thus the dif. equation

$$(4t^3+3y)dt+(3t+4y^3)dy=0$$

is exact.

(b) Based on the integrating factor from (a), find the general solution to the differential equation.

$$\begin{cases}\Psi_t=4t^3+3y\Rightarrow\Psi=t^4+3y\cdot t+C(y)\\ \Psi_y=3t+4y^3\overset{?}{=}3t+C'(y)\Rightarrow C'(y)=4y^3\Rightarrow\end{cases}$$

$C(y)=y^4$. Whence

$$t^4+3t\cdot y+y^4=C$$

is the general solution in the implicit form.

Q3 (25 pts.) Determine whether

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} \sin(2t) \\ -2t\cos(t) \end{bmatrix}, \quad \mathbf{x}^{(2)}(t) = \begin{bmatrix} \sin(t) \\ -t \end{bmatrix}$$

are linearly independent vector-valued functions on the interval $I = (-\pi, \pi)$. Explain your answer.

Note that

$$W(t) = \begin{vmatrix} \sin(2t) & \sin(t) \\ -2t\cos(t) & -t \end{vmatrix} = -t\sin(2t) + 2\sin(t)\cos(t)\cdot t$$

$$= -t\sin(2t) + t\sin(2t) = 0 \text{ for all } t \in I.$$

Now assume that $\lambda \vec{x}^{(1)}(t) + \mu \vec{x}^{(2)}(t) = \vec{0}, \ \forall t \in I$.

Then

$$\begin{cases} \lambda \sin(2t) + \mu \sin(t) = 0 \\ \lambda(-2t\cos(t)) + \mu(-t) = 0 \end{cases}, \ \forall t \in I$$

Put $t = \pi/2 \in I \Rightarrow \mu = 0$.

$t = \pi/4 \in I \Rightarrow \lambda = 0$.

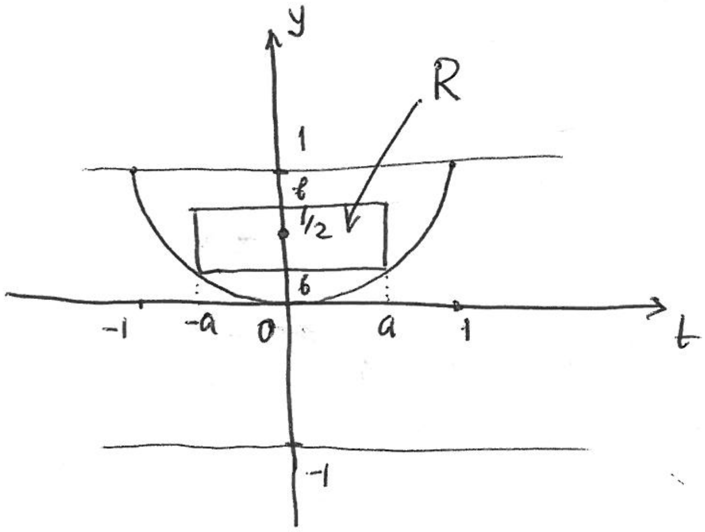
Thus $\vec{x}^{(1)}(t)$, $\vec{x}^{(2)}(t)$ are linearly independent vector-functions on I.

Q4 (25 pts.) Find and sketch a rectangular region R where conditions of Existence and Uniqueness Theorem are applicable to IVP $\begin{cases} y' = \left(t^2 - y\right)^{-1} \arcsin(y) \\ y(0) = 1/2 \end{cases}$.

Put $f(t,y) = \dfrac{\arcsin(y)}{t^2 - y}$

Then $\dfrac{\partial f}{\partial y} = \dfrac{\frac{1}{\sqrt{1-y^2}}(t^2-y) + \arcsin(y)}{(t^2-y)^2} = \dfrac{t^2-y + \arcsin(y)\sqrt{1-y^2}}{(t^2-y)^2\sqrt{1-y^2}}$

Choose $R = [-a, a] \times [1/2 - b, 1/2 + b]$.



Q5 (Bonus 10pts.) *a*) Suppose that a water is flowing (pumping) throughout a cylindrical pipe of a fixed radius d at a rate $r(t)$ into a tank. Find out how much water will be supplied after t minutes.

b) Consider a tank (initially) of 100 gal of water whose which contains 25 oz of salt dissolved in. The capacity of the tank is assumed to be large enough. Assume that water is pumping to the tank throughout a pipe of radius $3cm$ at a rate of $3t^2$ gal/ min. The entering water contains $1/2 + \sin(t)$ oz/gal of salt. Further, the well stirred mixture is draining out from the tank throughout the pipe of radius $5cm$ at a rate $2t$gal/min. Based on *a*) write down the related IVP to solve the problem. Note that the amount of water in the tank is changing versus t. **Don't solve the equation.**