M E T U
Northern Cyprus Campus

Math 219	Differential Equations	Final Exam	07.05.2013
Last Name Name : Student No	Dept./Sec.: Time : 16: 00 Duration ; 120 minutes	Signature	
6 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS	

1	2	3	4	5	6			

Q1 (15 p.) Find the general solution to the differential equation $\dfrac{dy}{dx} = \dfrac{2y^2 - x^2}{xy}$.

Let $\dfrac{y}{x} = v$ then, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx} = \dfrac{2v^2x^2 - x^2}{x.vx}$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v^2-1}{v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2-1}{v}$$

$$\Rightarrow \frac{v\,dv}{v^2-1} = \frac{dx}{x}, \quad v \neq 1, v \neq -1$$

(or $y \neq x$, $y \neq -x$)

(They are not sol's $y = x$, $y = -x$)

say $v^2 - 1 = u$

$v\,dv = \dfrac{du}{2}$

$$\frac{1}{2}\frac{du}{u} = \frac{dx}{x}$$

$$\frac{1}{2}\ln u = \ln x + C$$

$$\sqrt{u} = x\,e^C$$

$$\sqrt{v^2-1} = x.C_0$$

$$\Rightarrow \sqrt{\frac{y^2}{x^2} - 1} = x.C_0$$

$$y^2 - x^2 = C_1 x^4$$

Q2 (20 p.) Find the general solution to the following nonhomogeneous 2×2-linear system of differential equations

$$\mathbf{x}'(t) = \underbrace{\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}}_{A} \mathbf{x}(t) + \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix}, \quad t > 0.$$

$$\det(A - \lambda I) = \det \begin{vmatrix} 2-\lambda & -1 \\ 4 & -2-\lambda \end{vmatrix} = \lambda^2 - 4 + 4 = \lambda^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 0.$$

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} 2u_1 = u_2 \\ 4u_1 = 2u_2 \end{matrix}\right\} \Rightarrow \vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \left.\begin{matrix} 2v_1 - v_2 = 1 \\ 4v_1 - 2v_2 = 2 \end{matrix}\right\} \Rightarrow \vec{v} = \begin{bmatrix} s \\ 2s-1 \end{bmatrix} = s \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{in the direction of}} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

solutions; $x^1(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $x^2(t) = \begin{bmatrix} t \\ 2t-1 \end{bmatrix}$ $\left(\text{or } \Psi(t) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} t \\ 2t- \end{bmatrix} \right.$

$$\psi(t) = \begin{bmatrix} 1 & t \\ 2 & 2t-1 \end{bmatrix} \Rightarrow \psi^{-1}(t) = \begin{bmatrix} -2t+1 & t \\ 2 & -1 \end{bmatrix}$$

$$x(t) = \psi(t)\, c + \psi(t) \int_{t_0}^{t} \psi^{-1}(s) g(s)\, ds \quad \left(\text{or } \Psi(t)\, u'(t) = \begin{bmatrix} t^{-3} \\ t^{-2} \end{bmatrix} \right)$$

$$\Rightarrow \int \begin{bmatrix} -2t+1 & t \\ 2 & -1 \end{bmatrix} \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix} dt = \int \begin{bmatrix} -2t^{-2} + t^{-3} - t^{-1} \\ 2t^{-3} + t^{-2} \end{bmatrix} dt = \begin{bmatrix} 2t^{-1} - \frac{t^{-2}}{2} - \ln t \\ -t^{-2} - t^{-1} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 1 & t \\ 2 & 2t-1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 1 & t \\ 2 & 2t-1 \end{bmatrix} \begin{bmatrix} 2t^{-1} - \frac{t^{-2}}{2} - \ln t \\ -t^{-2} - t^{-1} \end{bmatrix}$$

$$x(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 2t-1 \end{bmatrix} + \begin{bmatrix} -1 + t^{-1} - \frac{t^{-2}}{2} - \ln t \\ -2 + 3t^{-1} - 2\ln t \end{bmatrix}$$

Q3 (20 p.) Find the solution to the IVP

$$\begin{cases} y^{(4)} - y = \delta(t-5), \\ y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0 \end{cases}$$

(Don't use the Convolution Theorem of Laplace).

$$\mathcal{L}\{y^{(4)} - y\} = \mathcal{L}\{\delta(t-5)\} \Rightarrow s^4 Y(s) - Y(s) = e^{-5s}$$

$$\Rightarrow Y(s) = \frac{e^{-5s}}{s^4-1} = \frac{e^{-5s}}{4}\left[\frac{1}{s-1} - \frac{1}{s+1} - \frac{2}{s^2+1}\right]$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{4} u_5(t) \cdot \left[e^{t-5} - e^{-t+5} - 2 \sin(t-5)\right]$$

Q4 (15 p.) Based on Separation of Variables technique, replace the partial differential equation $u_{xx} + u_{tt} + tu = 0$ be a pair of ordinary differential equations.

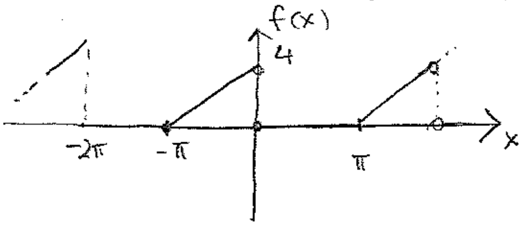
Let $u(x,t) = X(x)T(t)$, $u_{xx} = X''(x)T(t)$; $u_{tt} = X(x)T''(t)$

$$\Rightarrow X''(x)T(t) + X(x)T''(t) + t\,X(x)T(t) = 0.$$

$$X''(x)T(t) = -X(x)\left(T''(t) + t\,T(t)\right)$$

$$\frac{X''(x)}{X(x)} = -\frac{T''(t) + t\,T(t)}{T(t)} \overset{\text{say}}{=\!=} -\lambda \Rightarrow \begin{array}{l} X''(x) + \lambda X(x) = 0 \\ T''(t) + (\lambda - t)T(t) = 0 \end{array}$$

Q5 (15=8+3+4 p.) Find the Fourier series expansion $F(x)$ (or $S_f(x)$) of the function $f(x) = \begin{cases} \frac{4}{\pi}x + 4 & \text{if } -\pi \le t < 0 \\ 0 & \text{if } 0 \le t < \pi \end{cases}$, and then compute $F(17\pi) = ?$ and $F(-17) = ?$ (or $S_f(17\pi) = ?$ and $S_f(-17) = ?$).



$$S_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\pi} + b_n \sin\frac{n\pi x}{\pi}\right)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\cdot\frac{4\pi}{2} = 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{n\pi x}{\pi}\right)dx = \frac{1}{\pi}\int_{-\pi}^{0}\left(\frac{4x}{\pi}+4\right)\cos(nx)\,dx = \frac{1}{\pi}\left[\left(\frac{4x}{\pi}+4\right)\frac{\sin nx}{n} + \frac{4}{\pi}\frac{\cos nx}{n^2}\right]\Bigg|_{-\pi}^{0} = \frac{4}{\pi^2}(1-$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{n\pi x}{\pi}\right)dx = \frac{1}{\pi}\int_{-\pi}^{0}\left(\frac{4x}{\pi}+4\right)\sin(nx)\,dx = \frac{1}{\pi}\left[\left(\frac{4x}{\pi}+4\right)\cdot\frac{-\cos nx}{n} + \frac{4}{\pi}\frac{\sin(nx)}{n^2}\right]\Bigg|_{-\pi}^{0} = -\frac{4}{\pi n}$$

$$S_f(x) = 2 + \sum_{n=1}^{\infty}\left[\frac{4}{\pi^2}\left(\frac{1-\cos(n\pi)}{n^2}\right)\cdot\cos(nx) + \left(-\frac{4}{\pi}\right)\frac{\sin(nx)}{n}\right]$$

$$S_f(17\pi) = \frac{f(\pi_-) + f(\pi_+)}{2} = 0 \quad ; \quad S_f(-17) = S_f(6\pi - 17) = 0.$$

Q6 (15=7+8 p.) Compute the following convolutions: **a)** $t^{101} * t^{202} = ?$

$$\mathcal{L}\{t^{101}\} = \frac{101!}{s^{102}} \; ; \; \mathcal{L}\{t^{202}\} = \frac{202!}{s^{203}} \Rightarrow \mathcal{L}\{t^{101} * t^{202}\} = \frac{101!\cdot 202!}{s^{305}}$$

$$\mathcal{L}^{-1}\left\{\frac{101!\cdot 202!}{s^{305}}\right\} = \mathcal{L}^{-1}\left\{\frac{101!\,202!}{304!}\cdot\frac{304!}{s^{305}}\right\} = \frac{101!\,202!}{304!}\cdot t^{304}$$

b) $\delta(t-6) * f(t) = ?$, where $f(t)$ is a continuous function on $[0, \infty)$ whose growth is at most exponential.

$$\mathcal{L}\{\delta(t-6) * f(t)\} = e^{-6s}\cdot\mathcal{L}\{f(t)\}$$

$$\mathcal{L}^{-1}\{e^{-6s}\cdot\mathcal{L}\{f(t)\}\} = u_6(t)\, f(t-6)$$