M E T U
Northern Cyprus Campus

Math 219	Differential Equations	Final Exam	10.08.2012
Last Name Name : Student No KEY	Dept./Sec.: Time : 09: 00 Duration : 90 *minutes*		Signature
5 QUESTIONS ON 4 PAGES			TOTAL 100 POINTS

1	2	3	4	5				

Q1 (25 p.) Find the inverse Laplace transform $f(t)$ of the function $F(s) = \dfrac{e^{-13s}}{s^3-1} + e^{-15s}$ (Do not use the convolution operation).

First note that $s^3 - 1 = (s-1)(s^2+s+1) = (s-1)\left(\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}\right)$ and using the partial fraction expansion, we have

$$\frac{1}{s^3-1} = \frac{1}{3}\,\frac{1}{s-1} - \frac{1}{3}\,\frac{s+2}{s^2+s+1} = \frac{1}{3}\,\frac{1}{s-1} - \frac{1}{3}\,\frac{s+\frac{1}{2}+\frac{3}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{1}{3}\,\frac{1}{s-1} - \frac{1}{3}\,\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{\sqrt{3}}\,\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

By our second main shifting formular we derive that

$$f(t) = u_{13}(t)\,g(t-13) + \delta(t-15) \quad \text{with } G(s) = \frac{1}{s^3-1}$$

Using our first main shifting formular, we have

$$g(t) = \frac{1}{3}e^{t} - \frac{1}{3}e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

Thereby

$$f(t) = u_{13}(t)\left[\frac{1}{3}e^{t-13} - \frac{1}{3}e^{-\frac{1}{2}(t-13)}\cos\left(\frac{\sqrt{3}}{2}(t-13)\right) - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}(t-13)}\sin\left(\frac{\sqrt{3}}{2}(t-1\right.\right.$$

$$+ \delta(t-15)$$

Q2 (10 p.) Let $f(x), g(x) \in C[0,\infty)$, $c > 0$, and let $u_c(x)\, g(x-c)$ be the shifted function. Based on the definition of the convolution, and an elementary (Mat119) substitution, prove that $[f(x) * (u_c(x)\, g(x-c))](t) = [f(x) * g(x)](t-c)$ for all $t \geq c$.

By its very definition, we have

$$[f(x) * (u_c(x) g(x-c))](t) = \int_0^t f(t-\tau) u_c(\tau) g(\tau - c)\, d\tau = \int_c^t f(t-\tau) g(\tau-c)\, d\tau$$

$$\left| \begin{array}{l} u = \tau - c \\ du = d\tau \\ \tau = c \Rightarrow u = 0 \\ \tau = t \Rightarrow u = t-c \end{array} \right| = \int_0^{t-c} f(t-(u+c)) g(u)\, du = \int_0^{t-c} f(t-c-u) g(u)\, du$$

$$= (f * g)(t-c) \quad \text{for all } t \geq c.$$

Q3 (20 p.) Using the convolution theorem, find the solution to IVP:

$$\begin{cases} y^{(3)} - y'' + 2y = \cos(2t) + \delta(t-21) \\ y(0) = y'(0) = 0, \quad y''(0) = 1. \end{cases}$$

Apply Laplace transform to both sides of the dif. equation:

$$\mathcal{L}\{y^{(3)}\} - \mathcal{L}\{y''\} + 2\mathcal{L}\{y\} = \frac{s}{s^2+4} + e^{-21s}. \quad \text{But}$$

$$\mathcal{L}\{y^{(3)}\} = s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0) = s^3 Y(s) - 1,$$

$$\mathcal{L}\{y''\} = s^2 Y(s) - s y(0) - y'(0) = s^2 Y(s)$$

$$(s^3 - s^2 + 2) Y(s) = 1 + \frac{s}{s^2+4} + e^{-21s}. \quad \text{But } s^3 - s^2 + 2 =$$

$$= (s+1)(s^2 - 2s + 2) = (s+1)((s-1)^2 + 1). \quad \text{Whence}$$

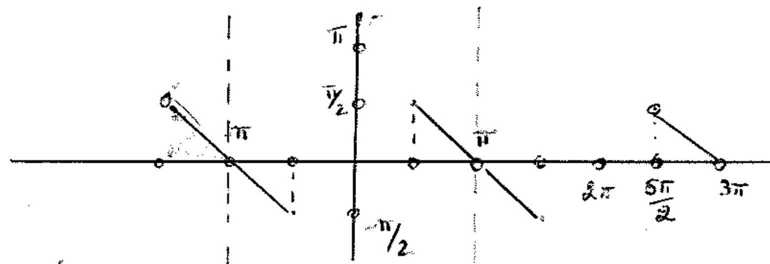
$$Y(s) = \frac{1}{(s+1)((s-1)^2+1)} + \frac{s}{s^2+4} \frac{1}{s+1} \frac{1}{(s-1)^2+1} + \frac{e^{-21s}}{(s+1)((s-1)^2+1)}$$

and

$$y(t) = e^{-t} * (e^t \sin(t)) + \cos(2t) * e^{-t} * (e^t \sin(t)) +$$

$$+ u_{21}(t) \left(e^{-t} * e^t \sin(t)\right)(t-21)$$

Q4 (20 p.) Consider the function $f(x) = \begin{cases} 0 & if \quad 0 < x < \pi/2; \\ \pi - x & if \quad \pi/2 < x < \pi; \end{cases}$. Find its sine Fourier series $S(x)$, and compute the values $S(5\pi/2)$ and $S(10)$ based on Fourier Convergence Theorem.



We extend $f(x)$ as an odd function to the interval $(-\pi, \pi)$.

Then $S(x) = \sum_{m=1}^{\infty} b_m \sin(mx)$

with $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\,dx = \frac{-2}{\pi n} \int_{\pi/2}^{\pi} (\pi - x) \cos'(nx)\,dx$

$$= \frac{-2}{\pi n} \left((\pi - x) \cos(nx) \Big|_{\pi/2}^{\pi} + \int_{\pi/2}^{\pi} \cos(nx)\,dx \right) =$$

$$= \frac{-2}{\pi n} \left(\frac{\pi}{2} \cos\left(\frac{\pi n}{2}\right) + \frac{1}{n} \sin(nx) \Big|_{\pi/2}^{\pi} \right) =$$

$$= \frac{-2}{\pi n} \left(\frac{\pi}{2} \cos\left(\frac{\pi n}{2}\right) - \frac{1}{n} \sin\left(\frac{\pi n}{2}\right) \right) =$$

$$= \begin{cases} n = 2k \Rightarrow -\frac{1}{n} \cos(\pi k) = \frac{(-1)^{k+1}}{2k} \\ n = 2k+1 \Rightarrow \frac{2}{\pi n^2} \sin\left((2k+1)\frac{\pi}{2}\right) = \frac{2(-1)^k}{\pi (2k+1)^2} \end{cases}$$

It follows that

$$S(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k} \sin(2kx) + \sum_{k=0}^{\infty} \frac{2(-1)^k}{\pi (2k+1)^2} \sin((2k+1)x)$$

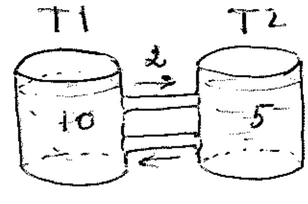
Finally, by Fourier Conv. Theorem,

$S\left(\frac{5\pi}{2}\right) = \frac{1}{2}\left(f\left(\frac{5\pi}{2}+\right) + f\left(\frac{5\pi}{2}-\right)\right) = \frac{\pi}{4}$ and

$S(10) = \frac{1}{2}\left(f(10+) + f(10-)\right) = f(10) = f(10 - 4\pi) = -10 + 4\pi$

for $-\pi < 10 - 4\pi < -\frac{\pi}{2}$

Q5 (25 p.) Consider two interconnected tanks $T1$ and $T2$ of salty water. $T1$ contains 10 gal of water and 1 oz of salt dissolved in it, whereas $T2$ contains 5 gal of water and 2 oz of salt in. The mixture is pumped from $T1$ into $T2$ at a rate of 2 gal/min by means of the first pipe connection and the mixture flows from $T2$ into $T1$ back at a rate of 2gal/min by means of the second pipe connection. Find the amounts of salt in both tanks at any time. What says your physical intuition on the amounts for a longer period time? Does the solution obtained confirm your intuition? (**Bonus 5p.** Predict the time t when salt amounts in both tanks will be exactly the same one).



Let $Q_k(t)$ be the amount of salt in the tank T_k, $k=1,2$.

We have

$$Q_1'(t) = 2\cdot\frac{Q_2(t)}{5} - 2\cdot\frac{Q_1(t)}{10} \quad \text{or} \quad \vec{Q}'(t) = \underbrace{\begin{bmatrix} -1/5 & 2/5 \\ 1/5 & -2/5 \end{bmatrix}}_{=A}\vec{Q}(t)$$

$$Q_2'(t) = 2\cdot\frac{Q_1(t)}{10} - 2\cdot\frac{Q_2(t)}{5}$$

with the initial condition $\vec{Q}(0) = \begin{bmatrix}1\\2\end{bmatrix}$, where $\vec{Q}(t) = \begin{bmatrix}Q_1(t)\\Q_2(t)\end{bmatrix}$

We have $\Delta(t) = \det(A-\lambda I) = \begin{vmatrix} -1/5-t & 2/5 \\ 1/5 & -2/5-t \end{vmatrix} =$

$= (1/5+t)(2/5+t) - 2/25 = t(t+3/5)$

$\sigma(A) = \{0, -3/5\}$

$\lambda = 0 \Rightarrow \begin{bmatrix} -1/5 & 2/5 \\ 1/5 & -2/5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow V_0 = \{x = 2y\},\ \vec{f}^{(1)} = \begin{bmatrix}2\\1\end{bmatrix}$

$\lambda = -3/5 \Rightarrow \begin{bmatrix} 2/5 & 2/5 \\ 1/5 & 1/5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow V_{-3/5} = \{x = -y\},\ \vec{f}^{(2)} = \begin{bmatrix}1\\-1\end{bmatrix}$

$\Psi(t) = Pe^{Jt} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-3t/5} \end{bmatrix} = \begin{bmatrix} 2 & e^{-3/5t} \\ 1 & -e^{-3/5t} \end{bmatrix}$

$\vec{Q}(t) = \Psi(t)\vec{c}$ – the general solution $\Rightarrow \begin{bmatrix}1\\2\end{bmatrix} = \Psi(0)\vec{c} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\vec{c}$

$\Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right] \Rightarrow \vec{Q}(t) = \Psi(t)\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix} 2 - e^{-3/5t} \\ 1 + e^{-3/5t} \end{bmatrix}$

$\Rightarrow Q_1(t) = 2 - e^{-3/5t},\ Q_2(t) = 1 + e^{-3/5t} \Rightarrow \lim_{t\to+\infty} Q_1(t) = 2,\ \lim_{t\to+\infty} Q_2(t)$.

Bonus: $Q_1(t) = Q_2(t) \Rightarrow 2 - e^{-3/5t} = 1 + e^{-3/5t} \Rightarrow \frac{1}{2} = e^{-3/5t} \Rightarrow$

$\Rightarrow -\ln(2) = -3/5\, t \Rightarrow t = \frac{5}{3}\ln(2)$