METU - NCC

	Introduction to Differential Equations **Final**	
Code : *MAT 219* Acad.Year: *2012-2013* Semester : *Fall* Date : *17.01.2013* Time : *16:00* Duration : *150 minutes*	Last Name: Name : KEY Department: Signature:	Student No.: Section:
	7 QUESTIONS ON 6 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6	7		

1. *(15 pts)* Find all real solutions of the following homogenous system, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$:

$$\mathbf{x}' = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \mathbf{x}$$

$\Delta(t) = \begin{vmatrix} 1-t & -2 \\ 2 & 1-t \end{vmatrix} = (1-t)^2 + 4 = (1-t-2i)(1-t+2i) = (t-(1+2i)) \times (t-(1-2i)) \Rightarrow \sigma(A) = \{1+2i, 1-2i\}$

$\lambda = 1+2i \Rightarrow A - \lambda = \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \sim \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}, V_{1+2i} =$

$= \{ix + y = 0\} = \text{Span}\{(1, -i)\}$. Thus

$\vec{z}(t) = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(1+2i)t}$ is a complex solution. But

$\vec{z}(t) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) \left(e^t \cos(2t) + i e^t \sin(2t) \right) =$

$= \begin{bmatrix} e^t \cos(2t) \\ e^t \sin(2t) \end{bmatrix} + i \begin{bmatrix} e^t \sin(2t) \\ -e^t \cos(2t) \end{bmatrix}$. Hence

$\Psi(t) = \begin{bmatrix} e^t \cos(2t) & e^t \sin(2t) \\ e^t \sin(2t) & -e^t \cos(2t) \end{bmatrix}, W(t) = -e^{2t},$

and $\vec{x}(t) = \Psi(t)\vec{c}$ is the general solution over $\mathbb{R}$.

2. *(15 pts)* Solve the initial value problem

$$\frac{dy}{dt} = y\sin t + 2te^{-\cos t}, \quad y(0) = 1$$

Let's use MVP: $y' = \sin(t) y \Rightarrow y_h = C e^{-\cos(t)}$.
Put $Y(t) = C(t) e^{-\cos(t)}$. Then $C'(t) e^{-\cos(t)} =$
$= 2t e^{-\cos(t)} \Rightarrow C'(t) = 2t \Rightarrow C(t) = t^2 \Rightarrow$
$\Rightarrow Y(t) = t^2 e^{-\cos(t)}$ is a special solution to
non-homg. eq. $\Rightarrow y = C e^{-\cos(t)} + t^2 e^{-\cos(t)}$.
IVP: $1 = y(0) = C e^{-1} \Rightarrow C = e$. Whence

$y = e^{1-\cos(t)} + t^2 e^{-\cos(t)}$ is the solution to IVP.

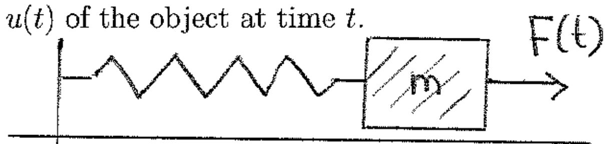
3. *(15 pts)* Find all solutions of the constant coefficient differential equation below:

$$y^{(4)} + 2y''' + 2y'' = 3e^t + t$$

First note that $\lambda^4 + 2\lambda^3 + 2\lambda^2 = \lambda^2(\lambda+1+i)(\lambda+1-i)$.
Therefore $y_h = C_1 + C_2 t + C_3 e^t \cos(t) + C_4 e^t \sin(t)$.
Based on MUC, we put
$Y(t) = t^2(At+B) + Ce^t$. Then
$Y'' = 2B + 6At + Ce^t$, $Y^{(3)} = 6A + Ce^t$, $Y^{(4)} = Ce^t \Rightarrow$
$\Rightarrow Ce^t + 12A + 2Ce^t + 4B + 12At + 2Ce^t =$
$= 5Ce^t + 12At + (4B + 12A) = 3e^t + t \Rightarrow C = \frac{3}{5}, A = \frac{1}{12},$
$B = -\frac{1}{4}$. Thus $Y(t) = -\frac{1}{4}t^2 + \frac{1}{12}t^3 + \frac{3}{5}e^t$ and
$y = y_h + Y$ is the general solution:
$y = C_1 + C_2 t + C_3 e^t \cos(t) + C_4 e^t \sin(t) - \frac{1}{4}t^2 + \frac{1}{12}t^3 + \frac{3}{5}e^t$.

4. *(5+10=15 pts)* Suppose that an object of mass $m = 3$ is attached to a spring with spring constant $\overset{=12}{k}$ whose other end is fixed onto a wall. The object moves horizontally on the table. There is a linear damping force and the damping constant is $\gamma = 12$. An external force is applied on the object in the form of a cosine function with amplitude $R = 9$ and frequency $\omega = 1$.

(a) Set up a differential equation for the position $u(t)$ of the object at time t.



$3u''(t) + 12u'(t) + 12u = F(t)$, where

$F(t) = 9\cos(t)$.

(b) Find the solution $u(t)$ if $u(0) = u'(0) = 0$.

Note that $u_h(t) = C_1 e^{-2t} + C_2 t e^{-2t}$, for $\lambda^2 + 4\lambda + 4 =$

$= (\lambda+2)^2 \Rightarrow \sigma(A) = \{-2^{(2)}\}$.

Put $U(t) = A\cos(t) + B\sin(t)$. Then

$(3A+4B)\cos(t) + (3B-4A)\sin(t) = 3\cos(t)$

$$\Rightarrow \begin{cases} 3A + 4B = 3 \\ -4A + 3B = 0 \end{cases} \Rightarrow A = \frac{9}{25},\ B = \frac{12}{25} \Rightarrow$$

$$\Rightarrow u(t) = C_1 e^{-2t} + C_2 t e^{-2t} + \frac{9}{25}\cos(t) + \frac{12}{25}\sin(t).$$

But $0 = u(0) = C_1 + \frac{9}{25} \Rightarrow C_1 = -\frac{9}{25}$, and

$$0 = u'(0) = -2C_1 + C_2 + \frac{12}{25} = \frac{30}{25} + C_2 = \frac{6}{5} + C_2$$

$$\Rightarrow C_2 = -\frac{6}{5} \Rightarrow$$

$$u(t) = -\frac{9}{25}e^{-2t} - \frac{6}{5}te^{-2t} + \frac{9}{25}\cos(t) + \frac{12}{25}\sin(t)$$

is the solution to IVP.

5. *(15 pts)* Find the solution to the initial value problem using the Laplace transform

$$y'' + 2y' + 3y = u_3(t)t + \delta(t-4), \quad y(0) = y'(0) = 0.$$

$$(s^2+2s+3)Y(s) = \mathcal{L}\{u_3(t)(t-3)\} + 3\mathcal{L}\{u_3(t)\} + e^{-4s} =$$

$$= \frac{e^{-3s}}{s^2} + \frac{3e^{-3s}}{s} + e^{-4s}$$

Note that

$$\frac{1}{s^2(s^2+2s+3)} = -\frac{2}{9s} + \frac{1}{3s^2} + \frac{1}{9}\,\frac{2(s+1)-1}{(s+1)^2+2}, \text{ and}$$

$$\frac{3}{s(s^2+2s+3)} = \frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+2}.$$ Therefore

$$Y(s) = \frac{7}{9}\,\frac{e^{-3s}}{s} + \frac{1}{3}\,\frac{e^{-3s}}{s^2} + \frac{e^{-3s}}{9}\,\frac{2(s+1)-1}{(s+1)^2+2} - e^{-3s}\frac{(s+1)+1}{(s+1)^2+2} +$$

$$+ \frac{e^{-4s}}{(s+1)^2+2}.$$ But

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{9}\,\frac{2(s+1)-1}{(s+1)^2+2}\right\} = \frac{2}{9}u_3(t)e^{-(t-3)}\cos(\sqrt{2}(t-3)) - \frac{1}{9}u_3(t)e^{-(t-3)} \times$$

$$\times \frac{1}{\sqrt{2}}\sin(\sqrt{2}(t-3));$$

$$\mathcal{L}^{-1}\left\{e^{-3s}\frac{(s+1)+1}{(s+1)^2+2}\right\} = u_3(t)e^{-(t-3)}\cos(\sqrt{2}(t-3)) + u_3(t)e^{-(t-3)} \times$$

$$\times \frac{1}{\sqrt{2}}\sin(\sqrt{2}(t-3));$$

$$\mathcal{L}^{-1}\left\{e^{-4s}\frac{1}{(s+1)^2+2}\right\} = u_4(t)e^{-(t-4)}\frac{1}{\sqrt{2}}\sin(\sqrt{2}(t-4)).$$

Hence

$$y(t) = \frac{7}{9}u_3(t) + \frac{1}{3}u_3(t)(t-3) + \frac{2}{9}u_3(t)e^{-(t-3)}\cos(\sqrt{2}(t-3))$$

$$-\frac{1}{9}u_3(t)e^{-(t-3)}\frac{1}{\sqrt{2}}\sin(\sqrt{2}(t-3)) - u_3(t)e^{-(t-3)}\cos(\sqrt{2}(t-3))$$

$$-u_3(t)e^{-(t-3)}\frac{1}{\sqrt{2}}\sin(\sqrt{2}(t-3)) + u_4(t)e^{-(t-4)}\frac{1}{\sqrt{2}}\sin(\sqrt{2}(t-4)).$$

6. *(15 pts)* Find the solution of the heat equation on a metal bar of conductivity $\alpha = 1$ and length $L = \pi$ such that

$$u(0,t) = u(\pi,t) = 0 \quad \text{for } t > 0,$$

$$u(x,0) = \frac{1}{3}\sin(3x) - \frac{2}{5}\sin(5x)$$

We deal with the problem

$$\begin{cases} u_{xx} = u_t, & 0 < x < \pi, \ t > 0 \\ u(x,0) = \frac{1}{3}\sin(3x) - \frac{2}{5}\sin(5x) = f(x) \\ u(0,t) = u(\pi,t) = 0, & t > 0 \end{cases}$$

whose solution can be written as the series $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t} \sin(nx)$,

where $f(x) = \sum_{n=1}^{\infty} c_n \sin(nx)$ is the sine Fourier series expansion of $f(x)$.

But $c_3 = \frac{1}{3}$, $c_5 = -\frac{2}{5}$ and $c_n = 0, n \neq 3,5$ due to the uniqueness property of Fourier series expansion. Whence

$$u(x,t) = \frac{1}{3} e^{-9t} \sin(3x) - \frac{2}{5} e^{-25t} \sin(5x).$$

7. *(10+5=15 pts)* (a) Find the Fourier series expansion of $f(x)$ which is a periodic function with period 2π and

$$f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ \pi, & 0 \leq x < \pi \end{cases}$$

$$S_f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(mx) + b_m \sin(mx) \quad \text{with}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \pi\, dx = \pi, \quad a_n = \frac{1}{\pi}\int_0^{\pi} \pi \cos(nx)\, dx = 0, \text{ and}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \pi \sin(nx)\, dx = -\frac{\cos(nx)}{n}\Big|_0^{\pi} = \frac{1-(-1)^n}{n}$$

$$= \begin{cases} \frac{2}{n} & \text{if } n\text{-odd} \\ 0 & \text{if } n\text{-even} \end{cases} \qquad \text{Hence}$$

$$S_f(x) = \frac{\pi}{2} + 2\sum_{m\text{-odd}} \frac{\sin(mx)}{m} =$$

$$= \frac{\pi}{2} + 2\sin(x) + \frac{2}{3}\sin(3x) + \frac{2}{5}\sin(5x) + \ldots$$

(b) What is the value of the sum of the Fourier series at $x = 5\pi$? What about at $x = 15$?

By Fourier Convergence Theorem,

$$S_f(x) = \frac{f(x+) + f(x-)}{2}, \quad \forall x \in \mathbb{R}. \text{ Therefore}$$

$$S_f(5\pi) = \frac{\pi}{2} \text{ and } S_f(15) = f(15) =$$

$$= f(15 - 4\pi + 4\pi) = f(15-4\pi). \text{ But}$$

$$0 < 15 - 4\pi < \pi \Rightarrow f(15-4\pi) = \pi \Rightarrow S_f(15) = \pi$$