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Math 219 Midterm #2 Review (Fall 2011 - Benjamin Walter)

§3.7 & 3.8 Unforced and forced Vibrations

m = mass ← inertia.
γ = dampening coefficient ← energy loss.
k = spring constant ← pull towards equilibrium.

Unforced

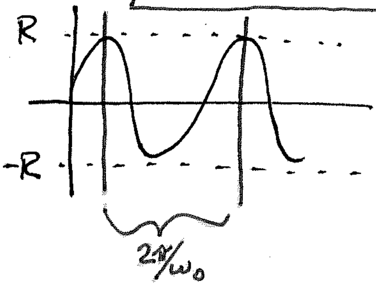
Undamped

$$my'' + ky = 0$$

natural freq $\omega_0 = \sqrt{k/m}$

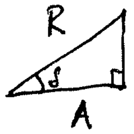
soln: $y = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

→ $y = R\cos(\omega_0 t - \delta)$



($R = \sqrt{A^2 + B^2}$

$\tan\delta = B/A$



)

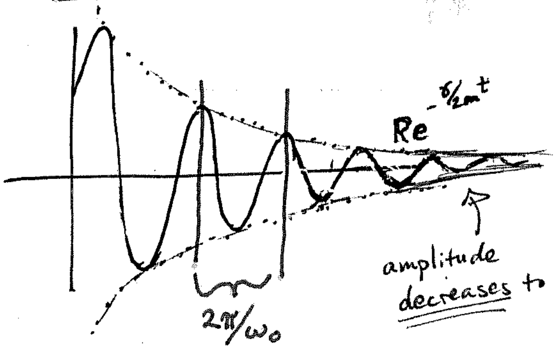
Damped

$$my'' + \gamma y' + ky = 0 \quad (\gamma^2 < 4mk)$$

natural freq $\omega_0 = \sqrt{k/m - (\gamma/2m)^2}$

soln: $y = Ae^{-\frac{\gamma}{2m}t}\sin(\omega_0 t) + Be^{-\frac{\gamma}{2m}t}\cos(\omega_0 t)$

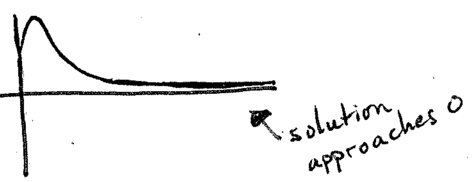
→ $y = Re^{-\frac{\gamma}{2m}t}\cos(\omega_0 t - \delta)$



Overdamped

$$my'' + \gamma y' + ky = 0 \quad (\gamma^2 > 4mk)$$

soln: $y = Ae^{r_1 t} + Be^{r_2 t}$



$r_1, r_2 = \frac{-\gamma}{2m} \pm \sqrt{(\frac{\gamma}{2m})^2 - k/m}$ (both negative)

Forced

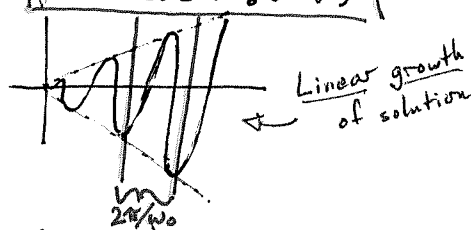
Undamped

$$my'' + ky = F\cos(\omega_F t)$$

$\omega_0 = \sqrt{k/m}$ forcing freq $= \omega_F$

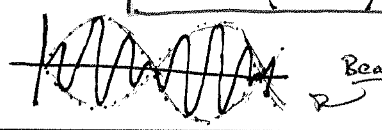
Resonance: if $\omega_0 = \omega_F$

$y \approx Rt\cos(\omega_0 t - \delta)$



Beats (no resonance): if $\omega_0 \neq \omega_F$

$y \approx R\sin\left(\left(\frac{\omega_F - \omega_0}{2}\right)t\right)\cdot\sin\left(\left(\frac{\omega_F + \omega_0}{2}\right)t\right)$

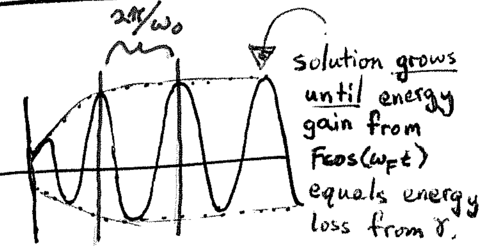


Damped

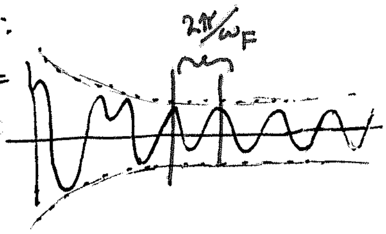
$$my'' + \gamma y' + ky = F\cos(\omega_F t) \quad (\gamma^2 < 4mk)$$

$\omega_0 = \sqrt{k/m - (\gamma/2m)^2}$, ω_F

Resonance if $\omega_0 = \omega_F$ (γ small)



No resonance: if $\omega_0 \neq \omega_F$



Overdamped (forced): Not Interesting

② **Important Laplace Transforms:** (Fall 2011 - Benjamin Walter)

$t^n \overset{\mathcal{L}}{\rightsquigarrow} \frac{n!}{s^{n+1}}$

$\cos bt \overset{\mathcal{L}}{\rightsquigarrow} \frac{s}{s^2+b^2}$

$\sin bt \overset{\mathcal{L}}{\rightsquigarrow} \frac{b}{s^2+b^2}$

$e^{at} f(t) \overset{\mathcal{L}}{\rightsquigarrow} F(s-a)$

$u_c \cdot f(t-c) \overset{\mathcal{L}}{\rightsquigarrow} e^{-cs} F(s)$

$\delta(t-c) \overset{\mathcal{L}}{\rightsquigarrow} e^{-cs}$

$y \overset{\mathcal{L}}{\rightsquigarrow} Y$

$y' \overset{\mathcal{L}}{\rightsquigarrow} sY - y(0)$

$y'' \overset{\mathcal{L}}{\rightsquigarrow} s^2Y - sy(0) - y'(0)$

$y''' \overset{\mathcal{L}}{\rightsquigarrow} s^3Y - s^2y(0) - sy'(0) - y''(0)$

etc

Computing $\mathcal{L}\{f\}$:
- use def: $\mathcal{L}\{f\} = \int_0^\infty e^{-st} \cdot f(t)\,dt$
- use $\frac{d}{dt}$: $\mathcal{L}\{\frac{d}{dt}f\} = s\mathcal{L}\{f\} - f(0)$
- use memorized formulas

Note:
- $\mathcal{L}^{-1}\{F(s)\} = e^{at}\mathcal{L}^{-1}\{F(s+a)\}$ EX: $\mathcal{L}^{-1}\{\frac{s}{(s+1)^2-1}\} = e^{-t} \cdot \mathcal{L}^{-1}\{\frac{(s-1)}{((s-1)+1)^2-1}\}$ ($=s$)
- $\mathcal{L}\{u_c \cdot f(t)\} = e^{-cs}\mathcal{L}\{f(t+c)\}$ EX: $\mathcal{L}\{u_2 \cdot t^2\} = e^{-2s}\mathcal{L}\{(t+2)^2\}$

Compute the following Laplace transforms: Example problems

basic:
- $\mathcal{L}\{e^{2t}\sin 3t\}$
- $\mathcal{L}\{4\}$
- $\mathcal{L}\{e^{4t}t^3\}$
- $\mathcal{L}\{u_2\}$
- $\mathcal{L}\{1 + (t-1)\cdot u_1 + \sin 2(t-2)\cdot u_2\}$

medium:
- $\mathcal{L}\{e^{2t+3}\cos t\}$
- $\mathcal{L}\{e^{t-1}t^2\}$
- $\mathcal{L}\{1 + tu_1 + t^2u_2\}$

hard:
- $\mathcal{L}\{e^t\cos t \cdot u_1\}$
- $\mathcal{L}\{e^{-t+2}\sin 3(t-4)\,u_4\}$
- $\mathcal{L}\{e^{3t+1}t^3 \cdot u_2\}$

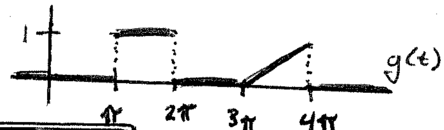
very hard:
- $\mathcal{L}\{t^2\cos 2t\}$
- $\mathcal{L}\{te^t\cos t\}$

→ solve these by computing derivatives and using $\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$

Step Functions

- $\mathcal{L}\{g(t)\}$ where $g(t) = \begin{cases} 1+t & t<1 \\ 2\cos(t-1) & 1 \le t < 1+\frac{\pi}{2} \\ 0 & 1+\frac{\pi}{2} \le t < \pi \\ -(t-\pi)^2 & \pi \le t \end{cases}$

- $\mathcal{L}\{g(t)\}$ where $g(t)$ is



$g(t) = (\text{start}) + (\text{to} - \text{from})\cdot u_{\text{when}} + \ldots$

Compute the following inverse Laplace transforms: Example problems

basic:
- $\mathcal{L}^{-1}\{\frac{s-1}{s^2-2s+5}\}$
- $\mathcal{L}^{-1}\{\frac{3}{(s-2)^2}\}$
- $\mathcal{L}^{-1}\{\frac{2}{s^2+4}e^{-3s}\}$

medium:
- $\mathcal{L}^{-1}\{\frac{s+1}{s(s^2+2)}\}$
- $\mathcal{L}^{-1}\{\frac{s-1}{s^2-2s+5}e^{-3s}\}$
- $\mathcal{L}^{-1}\{\frac{s+6}{(s-2)(s+2)}\}$
- $\mathcal{L}^{-1}\{\frac{3}{(s-2)^2}e^{-3s}\}$
- $\mathcal{L}^{-1}\{\frac{s}{s^2-2s+5}\}$
- $\mathcal{L}^{-1}\{\frac{2s^2+3s-1}{s^3}e^{-3s}\}$
- $\mathcal{L}^{-1}\{\frac{2s^2+3s-1}{(s-1)^3}\}$

(Note: You could use Taylor's Thm for this problem)

hard:
- $\mathcal{L}^{-1}\{\frac{s+2}{s^2-2s+5}e^{-3s+2}\}$
- $\mathcal{L}^{-1}\{\frac{2s-3}{s^2-3s+2}e^{-4s+2}\}$

medium

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Convolutions:

$$f * g = \mathcal{L}^{-1}\{\mathcal{L}\{f\}\cdot\mathcal{L}\{g\}\} = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Important Property:

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\cdot\mathcal{L}\{g\}$$

Example problems:

- $1 * t$
- $t * \sin t$
- $t * \cos t$
- $1 * u_2$
- $u_2 * u_3$
- $t^2 * t^3$

Recall: $ay'' + by' + cy = g(t)$ $\quad y(0)=0$, $y'(0)=0$

has solution $y = (\text{Impulse Response}) * g(t)$

where Impulse Response $= \mathcal{L}^{-1}\left\{\dfrac{1}{as^2+bs+c}\right\}$, and $\dfrac{1}{as^2+bs+c}$ is the "Transfer Function"

→ Differential Equation Solution is always

$$y = (\text{Homogeneous Soln.}) + (\text{Impulse Resp.}) * (\text{Forcing Funct.})$$

Putting it all together

Example problems:

- $y''' + 3y'' + 3y' + y = 0$ $\quad y(0)=1,\ y'(0)=2,\ y''(0)=3$
- $y'' - y' - 6y = \delta(t-3)$ $\quad y(0)=2,\ y'(0)=1$
- $y'' - 6y' + 9y = 2te^{-3t}$ $\quad y(0)=1,\ y'(0)=0$