METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES Final	
Code : *MAT 120* Acad.Year: *2014-2015* Semester : *SPRING* Date : *2.06.2015* Time : *16:00* Duration : *150 min*	Last Name: Name : Student # : Signature : 7 QUESTIONS ON 7 PAGES TOTAL 100 POINTS

1. (10)	2. (15)	3. (15)	4. (15)	5. (15)	6. (15)	7. (15)		

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(5+5=10 pts)* Compute the following limits.

(A) $\displaystyle\lim_{n\to\infty} \frac{\tan^{-1}(n^2+1)}{n^2+1} = 0$

$-\frac{\pi}{2} < \tan^{-1}(x) < \frac{\pi}{2}$, so

$$\frac{-\frac{\pi}{2}}{n^2+1} < \frac{\tan^{-1}(n^2+1)}{n^2+1} < \frac{\frac{\pi}{2}}{n^2+1}$$

by Squeeze Thm

as $n\to\infty$ as $n\to\infty$

0

(B) $\displaystyle\lim_{n\to\infty} \frac{\ln(\sin\frac{1}{n})}{\ln(n^2)} = \lim_{x\to\infty} \frac{\ln(\sin(\frac{1}{x}))}{\ln(x^2)} \left(\frac{-\infty}{\infty}\right) \overset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{\frac{\cos(\frac{1}{x})\cdot -\frac{1}{x^2}}{\sin(\frac{1}{x})}}{2\cdot\frac{1}{x}}$

$\ln(x^2) = 2\ln x$

$$= \lim_{x\to\infty} \frac{-\cos(\frac{1}{x})\cdot\frac{1}{x}}{2\cdot\sin(\frac{1}{x})}$$

$$= \lim_{x\to\infty} -\frac{\cos(\frac{1}{x})}{2}\cdot \lim_{x\to\infty} \frac{\frac{1}{x}}{\sin(\frac{1}{x})} = -\frac{1}{2}\cdot 1 = -\frac{1}{2}.$$

$\displaystyle\lim_{x\to\infty} \frac{1}{x} = 0$

2. *(5+5+5=15 pts)* Determine whether the given series are convergent or divergent. **State the tests you use and show details.**

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{\frac{1}{n}}}{n^2+1}$

$$\frac{2^{\frac{1}{n}}}{n^2+1} \leq \frac{2}{n^2+1} < 2 \cdot \frac{1}{n^2}$$

$$\underbrace{\sum_{n=1}^{\infty} \frac{2^{\frac{1}{n}}}{n^2+1}}_{\text{convergent by Comparison Test}} < 2 \cdot \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{convergent by p-test } p=2>1.}$$

(B) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$

Alternating Series.

(i) $\lim \sin\left(\frac{1}{n}\right) = \sin\left(\lim \frac{1}{n}\right) = \sin(0) = 0$ ✓

(ii) $\sin\left(\frac{1}{n+1}\right) < \sin\left(\frac{1}{n}\right)$ $\left[f(x) = \sin\left(\frac{1}{x}\right),\ f'(x) = \cos\left(\frac{1}{x}\right) \cdot \left(\frac{-1}{x^2}\right) < 0 \text{ when } x \geq 1. \right.$

By Alternating Series Test, it is convergent.

(C) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(\ln n)^2}$

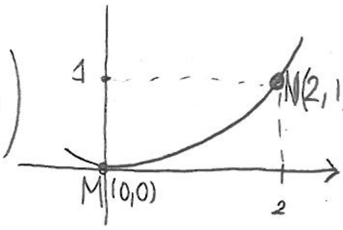
$$\lim \frac{n}{(\ln n)^2} = \lim_{x\to\infty} \frac{x}{(\ln x)^2} \left(\frac{\infty}{\infty}\right) \overset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{1}{2\ln x \cdot \frac{1}{x}} = \lim_{x\to\infty} \frac{x}{2\ln x} \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{1}{2 \cdot \frac{1}{x}} = \lim_{x\to\infty} \frac{x}{2} = +\infty.$$

By Test for Divergence (nth term test), it's divergent.

3. *(5+10=15 pts)* Let C be the segment of the parabola $4y = x^2$ joining $M(0,0)$ to $N(2,1)$.

(A) Evaluate $\int_C x\left(1+y\right) ds$. put $x=2t,\ y=t^2$, $0\le t\le 1$ $\left(\text{OR } \begin{array}{l} x=t \\ y=\frac{t^2}{2} \\ 0\le t\le 2\end{array}\right)$



$$\int_C x(1+y)\,ds = \int_0^1 2t\,(1+t^2)\sqrt{2^2+(2t)^2}\,dt$$

$$= \int_0^1 2t(1+t^2)\cdot 2(1+t^2)^{1/2}\,dt = \int_0^1 2t(1+t^2)^{3/2}\,dt \underset{\substack{u=1+t^2\\ du=2t\,dt}}{=} \int_1^2 u^{3/2}\,du$$

$$= \frac{2}{5}u^{5/2}\Big|_1^2 = \frac{2}{5}\left(\sqrt{32}-1\right)$$

(B) Find the potential function $\Psi = \Psi(x,y)$ such that

$$\left(\frac{x}{1+x^2+y^2}+y\right)\mathbf{i} + \left(\frac{y}{1+x^2+y^2}+x\right)\mathbf{j} = \nabla\Psi$$

and evaluate

$$\int_C \left(\frac{x}{1+x^2+y^2}+y\right)dx + \left(\frac{y}{1+x^2+y^2}+x\right)dy\,.$$

Note that $\frac{\partial}{\partial y}\left(\frac{x}{1+x^2+y^2}+y\right) = \frac{\partial}{\partial x}\left(\frac{y}{1+x^2+y^2}+x\right)$.

Hence, there exists $\Psi(x,y)$ such that
$$\Psi_x = \frac{x}{1+x^2+y^2}+y$$
$$\Psi_y = \frac{y}{1+x^2+y^2}+x$$

So, $\Psi(x,y) = \int\left(\frac{x}{1+x^2+y^2}+y\right)dx \underset{\text{using } u=1+x^2+y^2}{=} \frac{1}{2}\ln\left(1+x^2+y^2\right)+xy+C(y)$

$$\frac{y}{1+x^2+y^2}+x = \Psi_y = \frac{y}{1+x^2+y^2}+x+C'(y) \Rightarrow C'(y)=0 \Rightarrow C(y)=C$$

$$\therefore\ \Psi(x,y) = \frac{1}{2}\ln\left(1+x^2+y^2\right)+xy$$

$$\int_C\left(\frac{x}{1+x^2+y^2}+y\right)dx+\left(\frac{y}{1+x^2+y^2}+x\right)dy \underset{\text{Fund. Thm. of Line Int.}}{=} \Psi(x,y)\Big|_{(0,0)}^{(2,1)} = \Psi(2,1)-\Psi(0,0)$$

$$= \frac{\ln(6)}{2}+2$$

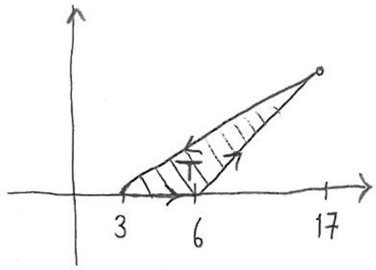
4. *(7+8=15 pts)*

(A) Evaluate

$$\int_C \left(7y + e^{x^2}\right) dx + \left(19x + e^{y^7}\right) dy$$

where C is the curve that traces out the boundary of the triangular region with vertices $K(3,0)$, $L(6,0)$ $M(17,5)$ in counterclockwise fashion.

$$\int_C \left(7y + e^{x^2}\right) dx + \left(19x + e^{y^7}\right) dy \underset{\text{by Green's Thm}}{=} \iint_T (19-7)\, dA$$



$$= \iint_T 12\, dA = 12 \cdot \frac{3 \cdot 5}{2} = 90$$

(B) Compute the area enclosed by the closed curve

$$\begin{aligned} x &= 2\cos(t) + \cos(2t) \\ y &= 2\sin(t) - \sin(2t) \end{aligned}$$

where $0 \leqslant t \leqslant 2\pi$.

$$\text{Area Enclosed} = \frac{1}{2} \int_C x\,dy - y\,dx$$

$$= \frac{1}{2} \int_0^{2\pi} \left[(2\cos t + \cos(2t))(2\cos t - 2\cos(2t)) - (2\sin t - \sin(2t))(-2\sin t - 2\sin(2t))\right] dt$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\left[4\cos^2 t - 2\cos t \cos(2t) - 2\cos^2(2t)\right] - \left[-4\sin^2 t - 2\sin(t)\cdot\sin(2t) + 2\sin^2(2t)\right]\right) dt$$

$$= \frac{1}{2} \int_0^{2\pi} \left[4\cos^2 t + 4\sin^2 t - 2\cos^2(2t) - 2\sin^2(2t) - 2(\cos t \cdot \cos(2t) - \sin(t)\sin(2t))\right] dt$$

$$= \frac{1}{2} \int_0^{2\pi} (4 - 2 - 2\cos(3t))\, dt = \frac{1}{2}\left(2t - 2\frac{\sin(3t)}{3}\right)\Bigg|_0^{2\pi} = 2\pi.$$

5. *(10+5=15 pts)* **(A)** Find the **interval of convergence** of power series

$$\sum_{n=0}^{\infty}(-1)^n\frac{n}{4^{n+1}(n^2+3)}x^n$$

$$\lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{\frac{\cancel{(-1)^{n+1}}\cdot(n+1)\cdot \cancel{x^{n+1}}}{4^{n+2}\cdot((n+1)^2+3)}}{\frac{\cancel{(-1)^n}\cdot n\cdot \cancel{x^n}}{\cancel{4^{n+1}}\cdot(n^2+3)}}\right| = \lim |x| \frac{(n+1)\cdot(n^2+3)}{4\cdot n\cdot(n^2+2n+4)}$$

$$= \left[\frac{|x|}{4}\cdot \underbrace{\lim \frac{n^3+n^2+3n+3}{n^3+2n^2+4n}}_{=1}\right] < 1 \Rightarrow \frac{|x|}{4} < 1 \Rightarrow |x|<4$$

$$-4<x<4$$

Boundaries: $x=4$ $\displaystyle\sum_{n=0}^{\infty}(-1)^n\cdot\frac{n\cdot 4^n}{4^{n+1}\cdot(n^2+3)} = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{n}{4\cdot(n^2+3)}$ (Alternating Series)

(i) $\displaystyle\lim \frac{n}{4n^2+12} = 0$ (ii) $\displaystyle\frac{n+1}{4(n+1)^2+12} < \frac{n}{4n^2+12}$

Convergent by A.S.T.

$x=-4$ $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\cdot n\cdot(-4)^n}{4^{n+1}\cdot(n^2+3)} = \sum_{n=0}^{\infty}\frac{n}{4(n^2+3)}$

$\displaystyle\lim\frac{\frac{n}{4n^2+12}}{\frac{1}{n}} = \lim\frac{n^2}{4n^2+12} = \frac{1}{4}>0$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent

by Limit Comparison Test, $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\cdot n(-4)^n}{4^{n+1}(n^2+3)}$ is divergent, too.

Interval of Convergence $= (-4,4]$

(B) If $f(x)=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{n}{4^{n-1}(n^2+3)}x^n$, then compute $g^{(120)}(0)$ where $g(x)=x^2f(x)$.

$$g(x) = x^2\cdot f(x) = x^2\sum_{n=0}^{\infty}(-1)^n\frac{n}{4^{n+1}(n^2+3)}\cdot x^n = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot n\cdot x^{n+2}}{4^{n+1}\cdot(n^2+3)}$$

$\displaystyle\frac{g^{(120)}(0)}{120!} = C_{120}$ and C_{120} is the coefficient of $x^{118+2}=x^{120}$

Hence, for $n=118$, we get $\displaystyle C_{120} = \frac{(-1)^{118}\cdot(118)}{4^{118+1}\cdot(118^2+3)}$

$$\boxed{g^{(120)}(0) = \frac{120!\cdot 118}{4^{119}\cdot(118^2+3)}}$$

6. *(5+5+5=15 pts)* Find the Taylor/Power series for the following functions around the given centers.

(A) $f(x) = \dfrac{3}{4+x}$ around $a = 0$.

$$\frac{3}{4+x} = 3\cdot\frac{1}{4-(-x)} = \frac{3}{4}\,\frac{1}{\left(1-\left(\frac{-x}{4}\right)\right)} = \frac{3}{4}\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n$$

$$\left|-\frac{x}{4}\right| < 1$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n\cdot 3\cdot x^n}{4^{n+1}}$$

(B) $h(x) = \dfrac{\cos(x^2)-1}{x^2}$, $x \neq 0$ around $a = 0$.

$$\cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \Rightarrow \cos(x^2) = \sum_{n=0}^{\infty}\frac{(-1)^n (x^2)^{2n}}{(2n)!}$$

$$\cos(x^2) - 1 = \left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots\right) - 1 = \sum_{n=1}^{\infty}(-1)^n\frac{x^{4n}}{(2n)!}$$

$$\frac{\cos(x^2)-1}{x^2} = \frac{1}{x^2}\sum_{n=1}^{\infty}(-1)^n\frac{x^{4n}}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)^n\cdot x^{4n-2}}{(2n)!}$$

(C) $k(x) = \cos(x)$ around $a = \pi$.

$C_0 = k(\pi) = -\dfrac{1}{0!}$ | $k(x) = \cos x$

$C_1 = \dfrac{k'(\pi)}{1!} = 0$ | $k'(x) = -\sin x$

$C_2 = \dfrac{k''(\pi)}{2!} = \dfrac{1}{2!}$ | $k''(x) = -\cos x$

$C_3 = \dfrac{k'''(\pi)}{3!} = 0$ | $k'''(x) = \sin x$

$C_4 = \dfrac{k^{(4)}(\pi)}{4!} = -\dfrac{1}{4!}$ | $k^{(4)}(x) = \cos x$

$$C_n = \begin{cases} 0 & n = 2k+1 \\ \dfrac{(-1)^{k+1}}{(2k)!} & n = 2k \end{cases}$$

$$\cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-\pi)^{2n}}{(2n)!}$$

7. *(5+5+5=15 pts)* An elliptic paraboloid is given by the following equation $z = 1 + x^2 + 5y^2$.

(A) Find the equation of the plane which is tangent to this elliptic paraboloid at (2, 1, 10).

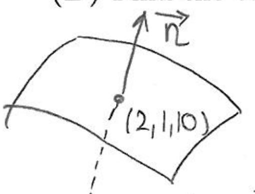
Tangent Plane Eqn: $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

$f_x = 2x$

$f_y = 10y$

$$\boxed{z - 10 = 4(x-2) + 10(y-1)}$$

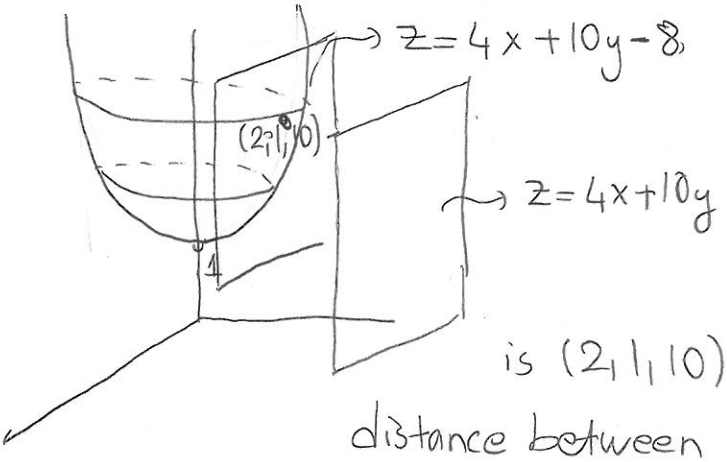
(B) Find the equation to the line which is perpendicular to this elliptic paraboloid at (2, 1, 10).



$\vec{n}$ is the direction of the line which is also normal of the tangent plane. Hence, $\vec{n} = \langle 4, 10, -1 \rangle$

$$\vec{r}(t) = \langle 2, 1, 10 \rangle + t \cdot \langle 4, 10, -1 \rangle.$$

(C) Find the minimum distance between the plane $z = 4x + 10y$ and this elliptic paraboloid.



Geometric Solution:

Since $z = 4x + 10y$ & $z = 4x + 10y - 8$ are parallel, the closest point on the ellipsoid to the plane is (2,1,10). Hence, it's enough to find the distance between (2,1,10) & $z = 4x + 10y$

Using distance formula, $d = \dfrac{|4 \cdot 2 + 10 \cdot 1 - 10|}{\sqrt{4^2 + 10^2 + (-1)^2}} = \dfrac{8}{\sqrt{117}}$

OR Using Lagrangen Multipliers: $d(x, y, z) = \dfrac{|4x + 10y - z|}{\sqrt{4^2 + 10^2 + (-1)^2}}$

subject to $1 + x^2 + 5y^2 - z = 0$