METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES MIDTERM 1	
Code : *MAT 120* Acad.Year: *2013-2014* Semester : *SPRING* Date : *05.04.2014* Time : *13:40* Duration : *110 min*	Last Name: Name : Student # : Solution Signature : 7 QUESTIONS ON 5 PAGES TOTAL 100 POINTS

1. (8)	2. (8)	3. (12)	4. (8)	5. (24)	6. (16)	7. (24)		

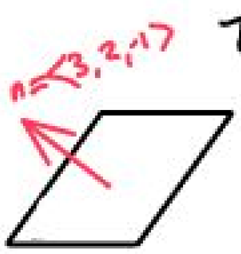
Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(4+4=8pts)* The following parts are about equations of lines and planes.

(a) Give the equation of the line through $(1,1,2)$ perpendicular to the plane $3x+2y-z=10$.

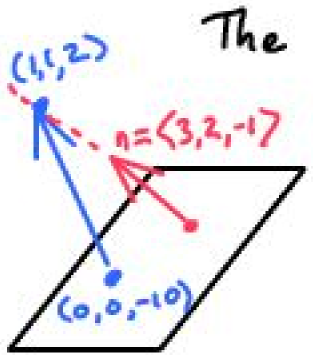
$3x+2y-z=0$ has normal direction $\underline{n}=\langle 3,2,-1\rangle$

The line through $(1,1,2)$ in direction $\underline{n}$ has formula:



$$\boxed{\begin{array}{l|l|l} \underline{r}(t)=\langle 1,1,2\rangle+\langle 3,2,-1\rangle t & x=1+3t & \\ \quad\;\;\; =\langle 1+3t,\ 1+2t,\ 2-t\rangle \quad \text{or} & y=1+2t \quad \text{or} & \frac{x-1}{3}=\frac{y-1}{2}=\frac{z-2}{-1} \\ & z=2-t & \end{array}}$$

(b) Compute the distance from $(1,1,2)$ to the plane $3x+2y-z=10$.



The plane $3x+2y-z=10$ has normal direction $\underline{n}=\langle 3,2,-1\rangle$

and contains the point $\left.\begin{array}{l}x=0\\y=0\end{array}\right\}\to 3\cdot 0+2\cdot 0-z=10$ $(0,0,-10)$

The vector from $(0,0,-10)$ to $(1,1,2)$ is $\langle 1,1,12\rangle$

(this is a vector from the plane to $(1,1,2)$)

The distance $\langle 1,1,12\rangle$ goes $\perp$ to the plane (parallel to $\langle 3,2,-1\rangle$)

is $\dfrac{|\langle 1,1,12\rangle\cdot\langle 3,2,-1\rangle|}{|\langle 3,2,-1\rangle|}=\dfrac{|3+2-12|}{\sqrt{14}}=\boxed{7/\sqrt{14} \text{ or } \sqrt{14}/2}$

2. *(4+4=8pts)* The following parts are about vector functions.

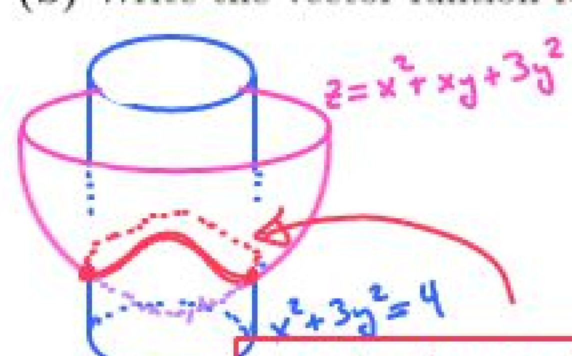
(a) Compute the tangent direction of $\mathbf{r}(t)=\langle t,\,t^2,\,t^3\rangle$ at the point $(2,4,8)$.

$\underline{r}(t)=\langle t,t^2,t^3\rangle$ The point $(2,4,8)$ is when $t=2$.

$\underline{r}'(t)=\langle 1,2t,3t^2\rangle$ At this point $\underline{r}'(2)=\langle 1,4,12\rangle$

$$\underline{T}=\frac{\langle 1,4,12\rangle}{\sqrt{1+16+144}}=\boxed{1/\sqrt{161}\,\langle 1,4,12\rangle}$$

(b) Write the vector funtion for the curve of intersection of $x^2+3y^2=4$ and $z=x^2+xy+3y^2$



First parameterize $x^2+3y^2=4$ $\begin{cases} x=2\cos t\\ \sqrt{3}y=2\sin t\end{cases}$

Plug in to get $z=x^2+xy+3y^2$

$=4\cos^2 t+4/\sqrt{3}\cos t\sin t+3\,4/3\sin^2 t$

$=4/\sqrt{3}\cos t\sin t+4$

$$\boxed{\underline{r}(t)=\langle 2\cos t,\ 2/\sqrt{3}\sin t,\ 4/\sqrt{3}\cos t\sin t+4\rangle}$$

③. *(3×4=12pts)* Find the given limits if they exist, or explain why they don't exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\,y^2}{x^6+y^2}$ Use the Squeeze Theorem:

$$0 \le \frac{y^2}{x^6+y^2} \le 1 \quad \text{if } (x,y)\neq(0,0)$$

so $$0 \le x^2\frac{y^2}{x^6+y^2} \le x^2 \quad \text{if } (x,y)\neq(0,0)$$

Since $\displaystyle\lim_{(x,y)\to(0,0)} 0 = 0$ and $\displaystyle\lim_{(x,y)\to(0,0)} x^2 = 0$, by the squeeze thm $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^6+y^2} = \boxed{0}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x\,y}{x^6+y^2}$ Consider the limit along different paths:

On the path $\left.\begin{matrix} y=0 \\ x\to 0\end{matrix}\right\}$ the limit is $\displaystyle\lim_{x\to 0}\frac{x\cdot 0}{x^6+0} = \lim_{x\to 0} 0 = \boxed{0}$

On the path $\left.\begin{matrix} y=x^3 \\ x\to 0\end{matrix}\right\}$ the limit is $\displaystyle\lim_{x\to 0}\frac{x\cdot(x^3)}{x^6+(x^3)^2} = \lim_{x\to 0}\frac{x^4}{x^6+x^6}$

$$= \lim_{x\to 0}\frac{x^4}{x^4}\frac{1}{2x^2} = \boxed{\infty}$$

$\neq$

These are not equal so limit does not exist

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2}$ Oops... this problem was supposed to be $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^4-4y^4}{x^2+2y^2}$ which equals 0 using continuity.

The problem as written has nonexisting limit because you can use the two paths

$\left.\begin{matrix} x=0 \\ y\to 0\end{matrix}\right\}$ $\displaystyle\lim_{y\to 0}\frac{-4y^2}{2y^2} = -2$ and $\left.\begin{matrix} y=0 \\ x\to 0\end{matrix}\right\}$ $\displaystyle\lim_{x\to 0}\frac{x^4}{x^2} = 0$

④ *(8pts)* Let $f(x,y) = \begin{cases} \frac{\sin(x^5+3y^2)}{x^4+y^2} & \text{if } (x,y)\neq(0,0), \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$ Find $f_x(0,0)$.

We will apply the definition of the derivative:

$$f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{\sin(h^5+0)}{h^4+0} - 0}{h}$$

$$= \lim_{h\to 0}\frac{\sin(h^5)}{h^5} = \boxed{1} \;!!!$$

5 . *(8+16=24pts)* The following parts are about tangent planes.

(a) Give the tangent plane to the surface $z = 2x^3 + xy - y^2$ at $x = 1$, $y = -2$. What is the normal vector ($\mathbf{n}$) of the tangent plane?

$z = f(x,y) = 2x^3 + xy - y^2$ $f(1,-2) = 2 - 2 - 4 = -4$

$f_x(x,y) = 6x^2 + y$ $f_x(1,-2) = 6 - 2 = 4$

$f_y(x,y) = x - 2y$ $f_y(1,-2) = 1 + 4 = 5$

Tangent Plane:

$z = f_x(1,-2)(x-1) + f_y(1,-2)(y+2) + f(1,-2)$

$$\boxed{z = 4(x-1) + 5(y+2) - 4} \rightarrow 0 = \underline{4}(x-1) + \underline{5}(y+2) \underline{-1}(z+4)$$

Normal vector:

$$\boxed{\underline{n} = \langle 4, 5, -1 \rangle}$$

(b) Suppose $g(s,t)$ is a function with $g(1,-2) = 2014$, $g_s(1,-2) = 3$ and $g_t(1,-2) = 5$. Let $f(x,y,z) = g(x^2 + yz,\ 2x - y^2 + 3z)$.

Find an equation for the tangent plane to the surface $f(x,y,z) = 2014$ at the point $(1,2,0)$

$f(x,y,z) = 2014$ is an implicitly defined surface

The tangent plane of an implicit surface is

$$f_x(1,2,0)(x-1) + f_y(1,2,0)(y-2) + f_z(1,2,0)(z-0) = 0$$

To compute the partial derivatives, use the chain rule:

$f = g(s,t)$ with $\begin{cases} s = x^2 + yz \\ t = 2x - y^2 + 3z. \end{cases}$

$s(1,2,0) = 1$
$t(1,2,0) = -2$ $f(1,2,0) = g(1,-2) = 2014$

$s_x = 2x$ $s_x(1,2,0) = 2$

$s_y = z$ $s_y(1,2,0) = 0$

$s_z = y$ $s_z(1,2,0) = 2$

$t_x = 2$ $t_x(1,2,0) = 2$

$t_y = -2y$ $t_y(1,2,0) = -4$

$t_z = 3$ $t_z(1,2,0) = 3$

$f_x = \partial f/\partial x = \partial g/\partial x = \partial g/\partial s \cdot \partial s/\partial x + \partial g/\partial t \cdot \partial t/\partial s$

$f_x(1,2,0) = 3 \cdot 2 + 5 \cdot 2 \leftarrow$
$= 16$

$f_y = \partial f/\partial y = \partial g/\partial y = \partial g/\partial s \cdot \partial s/\partial z + \partial g/\partial t \cdot \partial t/\partial y$

$f_y(1,2,0) = 3 \cdot 0 + 5 \cdot (-4) \leftarrow$
$= -20$

$f_z = \partial f/\partial z = \partial g/\partial z = \partial g/\partial s \cdot \partial s/\partial z + \partial g/\partial t \cdot \partial t/\partial z$

$f_z(1,2,0) = 3 \cdot 2 + 5 \cdot 3 \leftarrow$
$= 21$

Putting these values into the implicit tangent plane formula gives:

$$\boxed{16(x-1) - 20(y-2) + 21 \cdot z = 0}$$

⑥. *(5+3+8=16pts)* The following problems are about directional derivatives.

(a) Calculate the directional derivative $D_{\mathbf{u}}(x^2+3xy+y^2)$ in the direction of $\mathbf{u} = \langle 1, -1 \rangle$.

$$f = x^2 + 3xy + y^2$$

$$\left.\begin{array}{l} f_x = 2x + 3y \\ f_y = 3x + 2y \end{array}\right\} \nabla f = \langle 2x+3y, 3x+2y \rangle$$

$$D_{\langle 1,-1\rangle} f = \frac{\langle 1,-1\rangle \cdot \langle 2x+3y, 3x+2y\rangle}{|\langle 1,-1\rangle|}$$

$$= \frac{(2x+3y)-(3x+2y)}{\sqrt{2}}$$

$$= \boxed{1/\sqrt{2}\,(-x+y)}$$

(b) Calculate the double directional derivative $D_{\mathbf{u}}(D_{\mathbf{u}}(x^2+3xy+2y^2))$ where $\mathbf{u} = \langle 1, -1 \rangle$.

$$D_u D_u (x^2 + 3xy + 2y^2) = D_u\left(\frac{\langle 1,-1\rangle \cdot \langle 2x+3y, 3x+4y\rangle}{|\langle 1,-1\rangle|} \right)$$

$$= D_u\left(1/\sqrt{2}\,(-x-y) \right)$$

$$= 1/\sqrt{2}\ \frac{\langle 1,-1\rangle \cdot \langle -1,-1\rangle}{|\langle 1,-1\rangle|}$$

$$= 1/\sqrt{2} \cdot 1/\sqrt{2}\,(-1+1) = \boxed{0}$$

(c) Let $f(x, y)$ be differentiable and fix vectors $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$, $\mathbf{v} = \langle \frac{3}{5}, \frac{-4}{5} \rangle$, $\mathbf{w} = \langle -3, 1 \rangle$. If $D_{\mathbf{u}} f(x_0, y_0) = -1$ and $D_{\mathbf{v}} f(x_0, y_0) = 7$, find $D_{\mathbf{w}} f(x_0, y_0)$.

Let's solve for $\nabla f(x_0,y_0) = \langle a, b\rangle$

$$-1 = D_u f(x_0,y_0) = \frac{\langle 3/5, 4/5\rangle \cdot \langle a,b\rangle}{|\langle 3/5, 4/5\rangle|} = 3/5\,a + 4/5\,b$$

$$7 = D_v f(x_0,y_0) = \frac{\langle 3/5, -4/5\rangle \cdot \langle a,b\rangle}{|\langle -3/5, 4/5\rangle|} = 3/5\,a - 4/5\,b$$

$$-5 = 3a + 4b$$

$$+\ 35 = 3a - 4b$$

$$30 = 6a$$

$$a = 5,\ b = -5$$

$$\text{So } D_{\langle -3,1\rangle} f(x_0,y_0) = \frac{\langle -3,1\rangle \cdot \langle 5,-5\rangle}{|\langle -3,1\rangle|}$$

$$= \frac{-15-5}{\sqrt{10}}$$

$$= -20/\sqrt{10} = \boxed{-2\sqrt{10}}$$

7. *(12+12=24pts)* The following parts are about maxima and minima.

(a) Find and classify all critical points of the function $f(x, y) = x^3 + 3xy^2 + y^3 - 15y - 15x$.

Find critical points by solving $\nabla f = \langle 0, 0 \rangle$:

$$0 = f_x = 3x^2 + \cancel{3y^2} \cancel{-15}$$
$$-\quad 0 = f_y = 6xy + \cancel{3y^2} \cancel{-15}$$
$$0 = 3x^2 - 6xy$$
$$0 = 3x(x - 2y) \to x = 0 \text{ or } x = 2y$$

If $x = 0$ then
$$0 = f_y = 0 + 3y^2 - 15$$
$$y = \pm\sqrt{5}$$

If $x = 2y$ then
$$0 = f_y = 12y^2 + 3y^2 - 15$$
$$y = \pm 1$$

Critical points
$(0, \pm\sqrt{5})$
$(\pm 2, \pm 1)$

Classify critical points using the 2^{nd} derivative test.

$$f_{xx} = 6x \qquad f_{xy} = 6y$$
$$f_{yy} = 6x + 6y$$

$\boxed{(0, \sqrt{5}) : \text{saddle}}$
$f_{xx} = 0$
$f_{yy} = 6\sqrt{5}$
$f_{xy} = 6\sqrt{5}$
$D = 0 - 180 < 0$

$\boxed{(0, -\sqrt{5}) : \text{saddle}}$
$f_{xx} = 0$
$f_{yy} = -6\sqrt{5}$
$f_{xy} = -6\sqrt{5}$
$D = 0 - 180 < 0$

$\boxed{(2, 1) : \text{minimum}}$
$f_{xx} = 12 > 0$
$f_{yy} = 18 > 0$
$f_{xy} = 6$
$D = 12 \cdot 18 - 36 > 0$

$\boxed{(-2, -1) : \text{maximum}}$
$f_{xx} = -12 < 0$
$f_{yy} = -18 < 0$
$f_{xy} = -6$
$D = (-12)(-18) - 36 > 0$

(b) Find the maximum and minimum values of $f(x, y) = x^2 + 2x - y - y^2$ on the curve $\underbrace{x^2 - y^2}_{g} = 3$ **using the method of Lagrange multipliers**. (Çok dikkatli olun This problem is evil.)

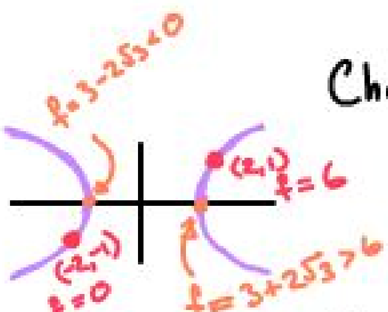
Lagrange multipliers: $\nabla f = \lambda \cdot \nabla g$

$$(\partial/\partial x)\quad (2x + 2 = \lambda \cdot 2x)\, y$$
$$(\partial/\partial y)\quad +\ (-1 - 2y = \lambda \cdot (-2y))\, x$$
$$(\cancel{2x}y + 2y) + (-x - \cancel{2xy}) = 0$$
$$2y - x = 0$$
$$2y = x$$

(plug into constraint)

$$x^2 - y^2 = 3$$
$$(2y)^2 - y^2 = 3$$
$$3y^2 = 3$$
$$y = \pm 1$$

$x = 2y$ so $y = 1 \Rightarrow x = 2$, $y = -1 \Rightarrow x = -2$



Check values of f:

$$f(2,1) = 4 + 4 - 1 - 1 = 6 \leftarrow \text{local min!}$$
$$f(-2,-1) = 4 - 4 + 1 - 1 = 0 \leftarrow \text{local max!}$$

See below...

Unfortunately, since $x^2 - y^2 = 3$ is not connected this does not mean that $(2,1)$ is max and $(-2,-1)$ is min....

In fact, checking $f(\pm\sqrt{3}, 0)$ shows that

- $(2,1)$ is a <u>local min</u> for its side of $x^2 - y^2 = 3$
- $(-2,-1)$ is a <u>local max</u> for its side of $x^2 - y^2 = 3$

(There is <u>no</u> global max or min)