the slope of $f''(x)$, we have

$$f'''(x) = 6$$

for all values of x. So f''' is a constant function and its graph is a horizontal line. Therefore, for all values of x,

$$f^{(4)}(x) = 0$$

We can also interpret the third derivative physically in the case where the function is the position function $s = s(t)$ of an object that moves along a straight line. Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

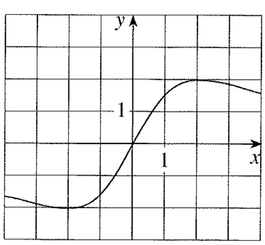
Thus the jerk j is the rate of change of acceleration. It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

We have seen that one application of second and third derivatives occurs in analyzing the motion of objects using acceleration and jerk. We will investigate another application of second derivatives in Section 3.3, where we show how knowledge of f'' gives us information about the shape of the graph of f. In Chapter 11 we will see how second and higher derivatives enable us to represent functions as sums of infinite series.

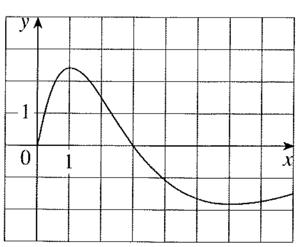
2.2 Exercises

1–2 Use the given graph to estimate the value of each derivative. Then sketch the graph of f'.

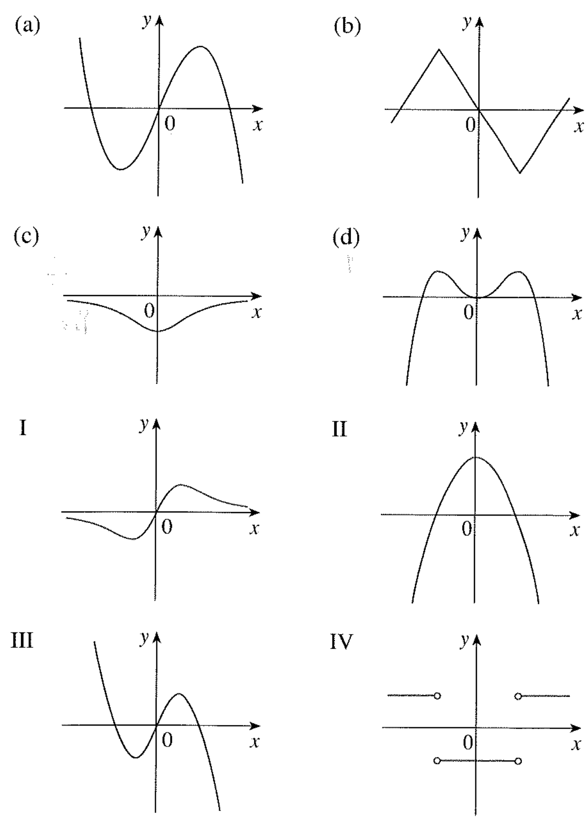
1. (a) $f'(-3)$ (b) $f'(-2)$ (c) $f'(-1)$
(d) $f'(0)$ (e) $f'(1)$ (f) $f'(2)$
(g) $f'(3)$



2. (a) $f'(0)$ (b) $f'(1)$ (c) $f'(2)$
(d) $f'(3)$ (e) $f'(4)$ (f) $f'(5)$
(g) $f'(6)$ (h) $f'(7)$

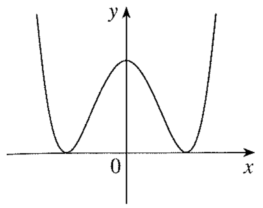


3. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for your choices.



4–11 Trace or copy the graph of the given function f. (Assume that the axes have equal scales.) Then use the method of Example 1 to sketch the graph of f' below it.

4.



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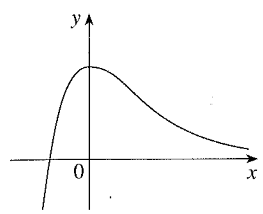

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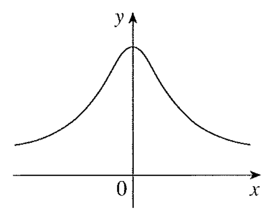

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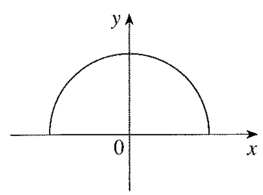

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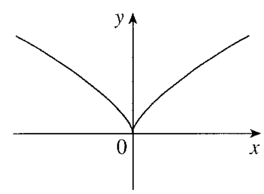

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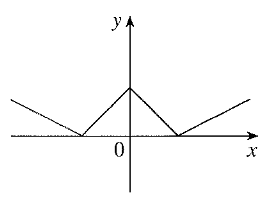

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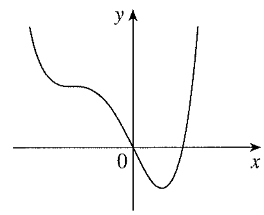

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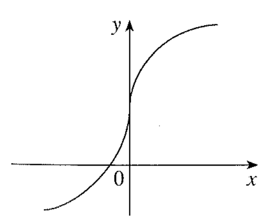
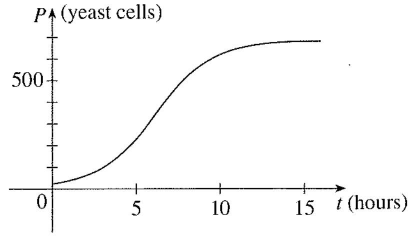


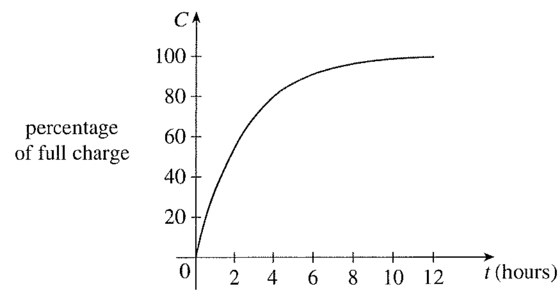
12. Shown is the graph of the population function $P(t)$ for yeast cells in a laboratory culture. Use the method of Example 1 to graph the derivative $P'(t)$. What does the graph of P' tell us about the yeast population?



13. A rechargeable battery is plugged into a charger. The graph shows $C(t)$, the percentage of full capacity that the battery reaches as a function of time t elapsed (in hours).

(a) What is the meaning of the derivative $C'(t)$?

(b) Sketch the graph of $C'(t)$. What does the graph tell you?

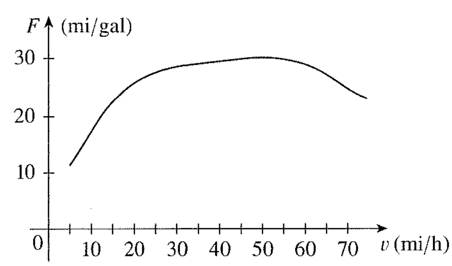


14. The graph (from the US Department of Energy) shows how driving speed affects gas mileage. Fuel economy F is measured in miles per gallon and speed v is measured in miles per hour.

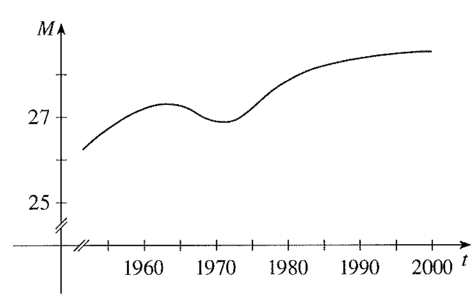
(a) What is the meaning of the derivative $F'(v)$?

(b) Sketch the graph of $F'(v)$.

(c) At what speed should you drive if you want to save on gas?



15. The graph shows how the average age of first marriage of Japanese men varied in the last half of the 20th century. Sketch the graph of the derivative function $M'(t)$. During which years was the derivative negative?



16. Make a careful sketch of the graph of the sine function and below it sketch the graph of its derivative in the same manner as in Exercises 4–11. Can you guess what the derivative of the sine function is from its graph?

17. Let $f(x) = x^2$.

(a) Estimate the values of $f'(0)$, $f'(\tfrac{1}{2})$, $f'(1)$, and $f'(2)$ by using a graphing device to zoom in on the graph of f.

(b) Use symmetry to deduce the values of $f'(-\tfrac{1}{2})$, $f'(-1)$, and $f'(-2)$.

(c) Use the results from parts (a) and (b) to guess a formula for $f'(x)$.

(d) Use the definition of derivative to prove that your guess in part (c) is correct.

18. Let $f(x) = x^3$.

(a) Estimate the values of $f'(0)$, $f'(\tfrac{1}{2})$, $f'(1)$, $f'(2)$, and $f'(3)$ by using a graphing device to zoom in on the graph of f.

(b) Use symmetry to deduce the values of $f'(-\tfrac{1}{2})$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.

(c) Use the values from parts (a) and (b) to graph f'.

(d) Guess a formula for $f'(x)$.

(e) Use the definition of derivative to prove that your guess in part (d) is correct.

19–29 Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

19. $f(x) = \tfrac{1}{2}x - \tfrac{1}{3}$

20. $f(x) = mx + b$

21. $f(t) = 5t - 9t^2$

22. $f(x) = 1.5x^2 - x + 3.7$

23. $f(x) = x^3 - 3x + 5$

24. $f(x) = x + \sqrt{x}$

25. $g(x) = \sqrt{9 - x}$

26. $f(x) = \dfrac{x^2 - 1}{2x - 3}$

27. $G(t) = \dfrac{1 - 2t}{3 + t}$

28. $f(x) = x^{3/2}$

29. $f(x) = x^4$

---

30. (a) Sketch the graph of $f(x) = \sqrt{6 - x}$ by starting with the graph of $y = \sqrt{x}$ and using the transformations of Section 1.3.

(b) Use the graph from part (a) to sketch the graph of f'.

(c) Use the definition of a derivative to find $f'(x)$. What are the domains of f and f'?

(d) Use a graphing device to graph f' and compare with your sketch in part (b).

31. (a) If $f(x) = x^4 + 2x$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

32. (a) If $f(x) = x + 1/x$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

33. The unemployment rate $U(t)$ varies with time. The table gives the percentage of unemployed in the Australian labor force measured at midyear from 1995 to 2004.

t	$U(t)$	t	$U(t)$
1995	8.1	2000	6.2
1996	8.0	2001	6.9
1997	8.2	2002	6.5
1998	7.9	2003	6.2
1999	6.7	2004	5.6

(a) What is the meaning of $U'(t)$? What are its units?

(b) Construct a table of estimated values for $U'(t)$.

34. Let $P(t)$ be the percentage of the population of the Philippines over the age of 60 at time t. The table gives projections of values of this function from 1995 to 2020.

t	$P(t)$	t	$P(t)$
1995	5.2	2010	6.7
2000	5.5	2015	7.7
2005	6.1	2020	8.9

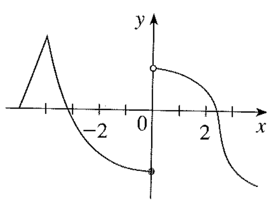
(a) What is the meaning of $P'(t)$? What are its units?

(b) Construct a table of estimated values for $P'(t)$.

(c) Graph P and P'.

35–38 The graph of f is given. State, with reasons, the numbers at which f is not differentiable.

35.



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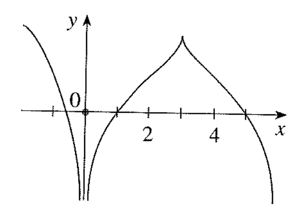

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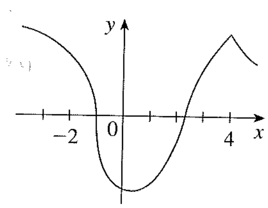

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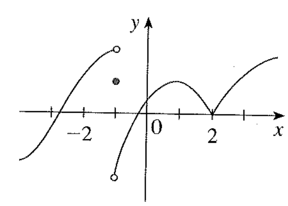

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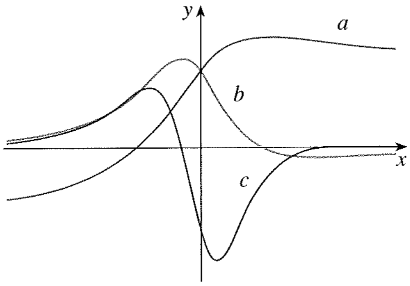
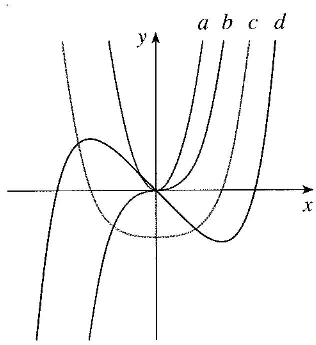
39. Graph the function $f(x) = x + \sqrt{|x|}$. Zoom in repeatedly, first toward the point $(-1, 0)$ and then toward the origin. What is different about the behavior of f in the vicinity of these two points? What do you conclude about the differentiability of f?

40. Zoom in toward the points $(1, 0)$, $(0, 1)$, and $(-1, 0)$ on the graph of the function $g(x) = (x^2 - 1)^{2/3}$. What do you notice? Account for what you see in terms of the differentiability of g.

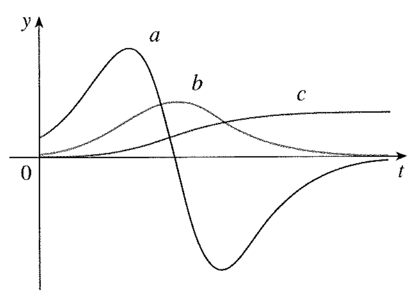
41. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.



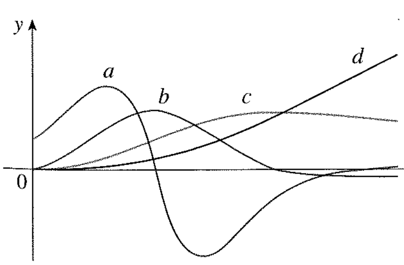
42. The figure shows graphs of f, f', f'', and f'''. Identify each curve, and explain your choices.



43. The figure shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve, and explain your choices.



44. The figure shows the graphs of four functions. One is the position function of a car, one is the velocity of the car, one is its acceleration, and one is its jerk. Identify each curve, and explain your choices.



45–46 Use the definition of a derivative to find $f'(x)$ and $f''(x)$. Then graph f, f', and f'' on a common screen and check to see if your answers are reasonable.

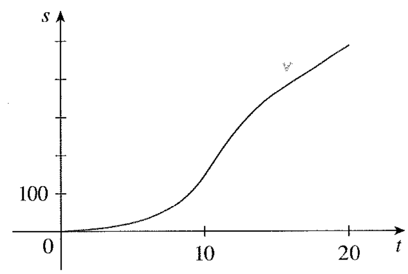
45. $f(x) = 3x^2 + 2x + 1$

46. $f(x) = x^3 - 3x$

---

47. If $f(x) = 2x^2 - x^3$, find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$. Graph f, f', f'', and f''' on a common screen. Are the graphs consistent with the geometric interpretations of these derivatives?

48. (a) The graph of a position function of a car is shown, where s is measured in meters and t in seconds. Use it to graph the velocity and acceleration of the car. What is the acceleration at $t = 10$ seconds?



(b) Use the acceleration curve from part (a) to estimate the jerk at $t = 10$ seconds. What are the units for jerk?

49. Let $f(x) = \sqrt[3]{x}$.
(a) If $a \neq 0$, use Equation 2.1.5 to find $f'(a)$.
(b) Show that $f'(0)$ does not exist.
(c) Show that $y = \sqrt[3]{x}$ has a vertical tangent line at $(0, 0)$. (Recall the shape of the graph of f. See Figure 13 in Section 1.2.)

50. (a) If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.
(b) If $a \neq 0$, find $g'(a)$.
(c) Show that $y = x^{2/3}$ has a vertical tangent line at $(0, 0)$.
(d) Illustrate part (c) by graphing $y = x^{2/3}$.

51. Show that the function $f(x) = |x - 6|$ is not differentiable at 6. Find a formula for f' and sketch its graph.

52. Where is the greatest integer function $f(x) = [\![x]\!]$ not differentiable? Find a formula for f' and sketch its graph.

53. (a) Sketch the graph of the function $f(x) = x|x|$.
(b) For what values of x is f differentiable?
(c) Find a formula for f'.

54. The **left-hand** and **right-hand derivatives** of f at a are defined by

$$f'_-(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$$

and
$$f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$

if these limits exist. Then $f'(a)$ exists if and only if these one-sided derivatives exist and are equal.
(a) Find $f'_-(4)$ and $f'_+(4)$ for the function

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ 5 - x & \text{if } 0 < x < 4 \\ \dfrac{1}{5-x} & \text{if } x \geqslant 4 \end{cases}$$

(b) Sketch the graph of f.
(c) Where is f discontinuous?
(d) Where is f not differentiable?

55. Recall that a function f is called *even* if $f(-x) = f(x)$ for all x in its domain and *odd* if $f(-x) = -f(x)$ for all such x. Prove each of the following.
(a) The derivative of an even function is an odd function.
(b) The derivative of an odd function is an even function.

56. When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running.
(a) Sketch a possible graph of T as a function of the time t that has elapsed since the faucet was turned on.
(b) Describe how the rate of change of T with respect to t varies as t increases.
(c) Sketch a graph of the derivative of T.

57. Let ℓ be the tangent line to the parabola $y = x^2$ at the point $(1, 1)$. The *angle of inclination* of ℓ is the angle ϕ that ℓ makes with the positive direction of the x-axis. Calculate ϕ correct to the nearest degree.

2.3 Differentiation Formulas

If it were always necessary to compute derivatives directly from the definition, as we did in the preceding section, such computations would be tedious and the evaluation of some limits would require ingenuity. Fortunately, several rules have been developed for finding derivatives without having to use the definition directly. These formulas greatly simplify the task of differentiation.

Let's start with the simplest of all functions, the constant function $f(x) = c$. The graph of this function is the horizontal line $y = c$, which has slope 0, so we must have $f'(x) = 0$. (See Figure 1.) A formal proof, from the definition of a derivative, is also easy:

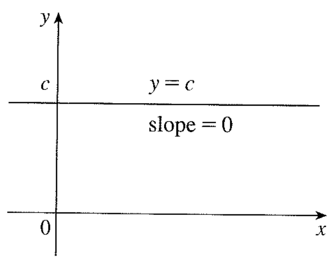


FIGURE 1
The graph of $f(x) = c$ is the line $y = c$, so $f'(x) = 0$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0$$

In Leibniz notation, we write this rule as follows.

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

Power Functions

We next look at the functions $f(x) = x^n$, where n is a positive integer. If $n = 1$, the graph of $f(x) = x$ is the line $y = x$, which has slope 1. (See Figure 2.) So

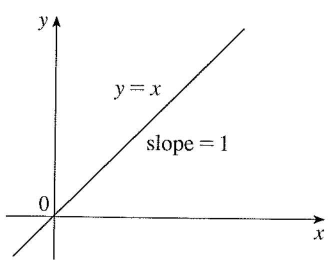


FIGURE 2
The graph of $f(x) = x$ is the line $y = x$, so $f'(x) = 1$.

1
$$\frac{d}{dx}(x) = 1$$