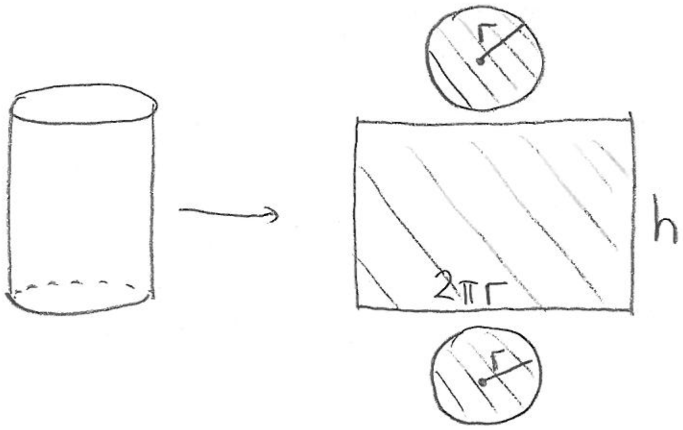
METU - NCC

CALCULUS with ANALYTIC GEOMETRY MIDTERM 2		
Code : *MAT 119* Acad.Year: *2014-2015* Semester : *FALL* Date : *13.12.2014* Time : *9:40* Duration : *120 min*	Last Name: Name : Student # : KEY Signature :	List #:
	5 QUESTIONS ON 6 PAGES TOTAL 100 POINTS	

1. (15)	2. (25)	3. (24)	4. (20)	5. (16)		

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(15pts)* A cylindirical container with volume $10\pi\ m^3$ will be produced. The material used for the top and the bottom lids of the container costs 5 TL per square meter and the rest of the material (used in the side panel) costs 8 TL per square meter. Find the dimensions of the cylinder which minimizes total material cost. JUSTIFY YOUR ANSWER.



$V(r,h) = \pi r^2 \cdot h = 10\pi\ m^3$

$h = \frac{10\pi}{\pi r^2} = \frac{10}{r^2}$

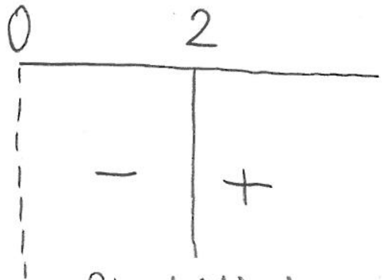
Cost will be: $C(r,h) = 2 \cdot \pi r^2 \cdot 5 + 2\pi r \cdot h \cdot 8 \qquad r, h > 0$

$C(r) = 10\pi r^2 + 16 \cdot \pi \cdot r \cdot \frac{10}{r^2}$

$= 10\pi\left(r^2 + \frac{16}{r}\right) \qquad r > 0$

$C'(r) = 10\pi\left(2r - \frac{16}{r^2}\right) = 10\pi \frac{(2r^3 - 16)}{r}$

$C'(r) = 0 \Rightarrow r^3 = 8 \Rightarrow r = 2$



$r = 2\ m$

$h = \frac{10}{2^2} = \frac{10}{4}\ m$

Global Min by
First Derivative Test for Absolute Extrema

2. *(4+4+6+6+5=25pts)* Given $f(x) = \dfrac{x^2-2}{(x-1)^2}$.

(a) Find the domain, x-intercepts and y-intercept of $f(x)$.

$\text{Dom}(f) = \mathbb{R} \setminus \{1\}$

<u>x-int:</u> $f(x) = 0 \Rightarrow x = \mp\sqrt{2}$, y-int: $x = 0 \Rightarrow y = -2$

(b) Find the asymptotes of $f(x)$.

Vertical Asymptote: $\lim\limits_{x\to 1^+} \dfrac{x^2-2}{(x-1)^2} = -\infty$, $\lim\limits_{x\to 1^-} \dfrac{x^2-2}{(x-1)^2} = -\infty$

Horizontal Asymptote: $\lim\limits_{x\to\infty} \dfrac{x^2-2}{x^2-2x+1} = \lim\limits_{x\to\infty} \dfrac{x^2(1-\frac{2}{x^2})}{x^2(1-\frac{2}{x}+\frac{1}{x^2})} = 1$

Similarly, $\lim\limits_{x\to -\infty} \dfrac{x^2-2}{x^2-2x+1} = 1$

(c) Find the intervals of increase/decrease and local max/min points of $f(x)$.

$$f'(x) = \frac{2x(x-1)^{\cancel{2}} - (x^2-2)\cdot 2\cdot\cancel{(x-1)}}{(x-1)^{\cancel{4}\,3}} = \frac{2x^2-2x-2x^2+4}{(x-1)^3}$$

$$= \frac{-2x+4}{(x-1)^3}$$

$f'(x) = 0 \Rightarrow x = 2$

$f'(x)$ doesn't exist at $x = 1$.

Intervals of Increase $= (1, 2)$

" " Decrease $= (-\infty, 1) \cup (2, +\infty)$

Local Max at $x = 2$

(d) Find the intervals of concavity and inflection points of $f(x)$.

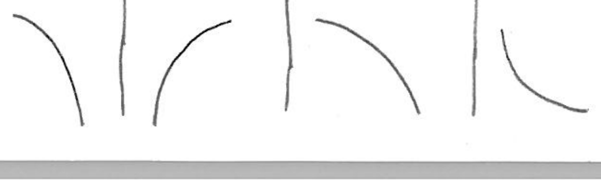
$$f''(x) = \frac{-2(x-1)^{\cancel{3}} - (-2x+4)\cdot 3\cancel{(x-1)^2}}{(x-1)^{\cancel{6}\,4}} = \frac{-2x+2+6x-12}{(x-1)^4}$$

$$= \frac{4x-10}{(x-1)^4}$$

$f''(x) = 0 \Rightarrow x = 5/2$

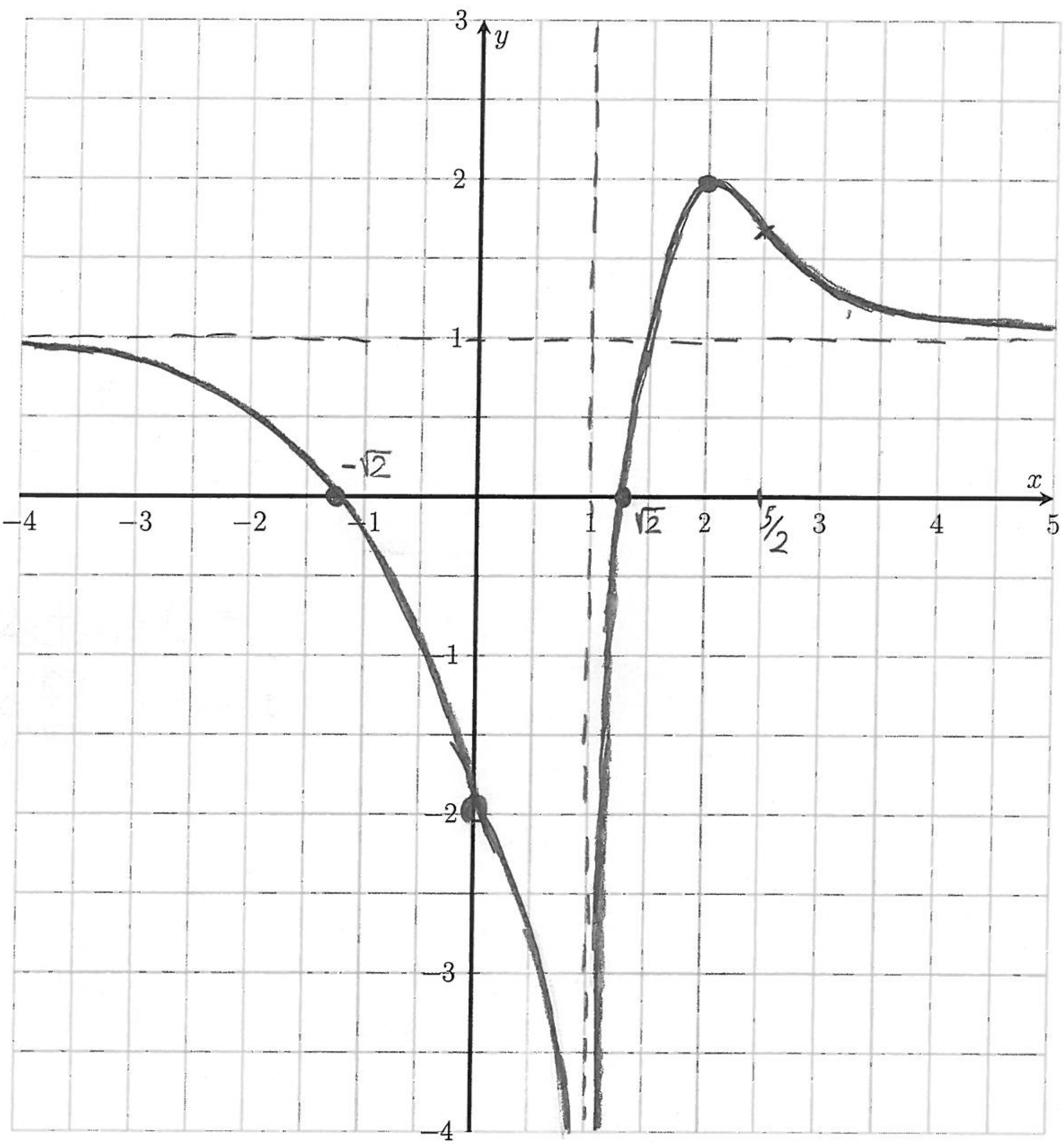
$f''(x)$ doesn't exist at $x = 1$

		1		2		5/2	
$f'(x)$	—	o	+	o	—		—
$f''(x)$	—	o	—		—	o	+
Shape of Curve							



$x = 5/2$ is an inflection point

(e) Sketch the graph of $f(x)$. **Don't forget** to indicate the intercepts, local maximum/minimum and inflection points on your graph, if there are any.



3. *(6×4=24pts)* Evaluate the following integrals.

(a) $\int (1-x^3)(\sqrt{x}+1)\,dx$

$$= \int \left(\sqrt{x} + 1 - x^3\cdot\sqrt{x} - x^3\right)dx = \int \left(x^{1/2} + 1 - x^{7/2} - x^3\right)dx$$

$$= \frac{2}{3}x^{3/2} + x - \frac{2}{9}x^{9/2} - \frac{x^4}{4} + C$$

(b) $\int_0^{\frac{\pi}{2}} \sqrt{1+3\sin x}\,\cos x\,dx$

$$\underset{\substack{u = 1+3\sin x \\ du = 3\cos x\,dx}}{=\!=\!=} \frac{1}{3}\int_1^4 \sqrt{u}\,du = \frac{1}{3}\cdot\frac{2}{3}u^{3/2}\Big|_1^4$$

$$= \frac{2}{9}\left(\sqrt{u}\right)^3\Big|_1^4 = \frac{16}{9} - \frac{2}{9} = \frac{14}{9}$$

(c) $\int \frac{x^5}{(1+x^2)^4}\,dx$

$$\underset{\substack{u = 1+x^2,\ x^2 = u-1 \\ du = 2x\,dx}}{=\!=} \int \frac{(x^2)^2\cdot x}{(1+x^2)^4}\,dx = \frac{1}{2}\int \frac{(u-1)^2}{u^4}\,du = \frac{1}{2}\int \frac{u^2-2u+1}{u^4}\,du$$

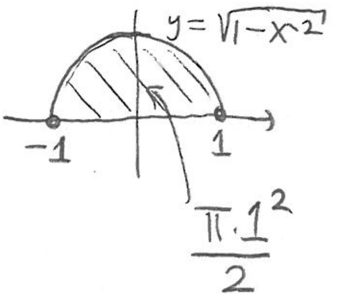
$$= \frac{1}{2}\int \left(u^{-2} - 2u^{-3} + u^{-4}\right)du$$

$$= \frac{1}{2}\left(\frac{-1}{u} + \frac{1}{u^2} - \frac{1}{3}\frac{1}{u^3}\right) + C$$

$$= \frac{1}{2}\left(\frac{-1}{1+x^2} + \frac{1}{(1+x^2)^2} - \frac{1}{3}\frac{1}{(1+x^2)^3}\right) + C$$

(d) $\int_{-1}^{1} (1-x)\sqrt{1-x^2}\,dx$

$$= \int_{-1}^{1} \sqrt{1-x^2}\,dx - \int_{-1}^{1} x\sqrt{1-x^2}\,dx = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$



$f(x) = x\sqrt{1-x^2}$

$f(-x) = -x\sqrt{1-(-x)^2}$

$= -x\sqrt{1-x^2} = -f(x)$

$f(x)$ is odd.

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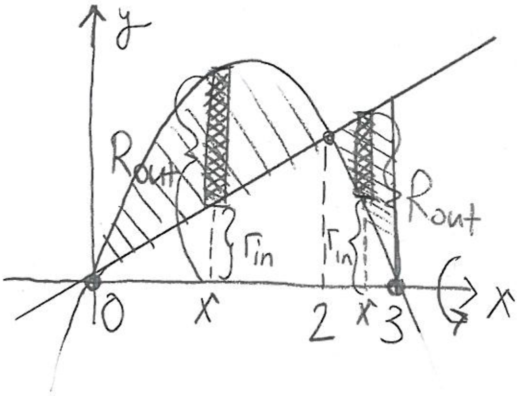
4. *(6+7+7=20pts)* Consider $f(x) = x$ and $g(x) = 3x - x^2$ on $0 \leqslant x \leqslant 3$.

(a) Write a definite integral which computes the area between f and g on $0 \leqslant x \leqslant 3$.

$$f(x) = g(x) \Rightarrow x = 3x - x^2$$

$$x^2 - 2x = 0 \Rightarrow x(x-2) = 0$$

$$x = 0 \text{ or } x = 2$$



$$\text{Area} = \int_0^2 [(3x-x^2) - x]\,dx + \int_2^3 [x - (3x-x^2)]\,dx$$

(b) Write a definite integral which computes the volume of the solid obtained by revolving the region bounded by f and g on $0 \leqslant x \leqslant 3$ around *x-axis*.

Cross-section = Washer

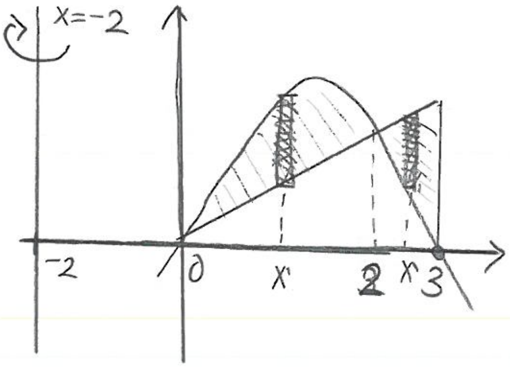
$0 \leqslant x \leqslant 2$: $R_{out} = (3x - x^2)$, $r_{in} = x$

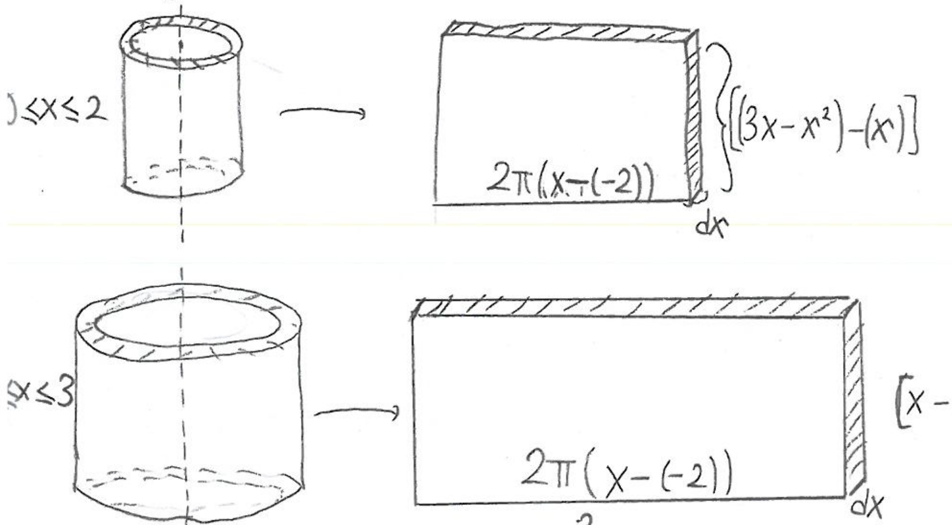
$2 \leqslant x \leqslant 3$: $R_{out} = x$, $r_{in} = (3x - x^2)$

$$\text{Volume} = \int_0^2 \pi(3x-x^2)^2 - \pi(x)^2\,dx + \int_2^3 \pi(x)^2 - \pi(3x-x^2)^2\,dx$$

(c) Write a definite integral which computes the volume of the solid obtained by revolving the region bounded by f and g on $0 \leqslant x \leqslant 3$ around $x = -2$.

Cylindrical Shell Method.





$$\text{Volume} = \int_0^2 2\pi(x+2)(2x-x^2)\,dx + \int_2^3 2\pi(x+2)(-2x+x^2)\,dx$$

5. *(8+8=16pts)* Two unrelated parts.

(a) Let $f(x) = \int_x^{x^2} g(t)\,dt$ and $g(t) = \int_1^{\sqrt{t}} h(u)du$. If $h(1) = 2$ then calculate $f''(1)$.

$$f'(x) = g(x^2)\cdot 2x - g(x)\cdot 1 \quad , \quad g'(t) = h(\sqrt{t})\cdot \frac{1}{2\sqrt{t}}$$

Fund. Thm. of Calculus Fund. Thm. of Calculus

$$f''(x) = g'(x^2)\cdot 2x\cdot 2x + g(x^2)\cdot 2 - g'(x)$$

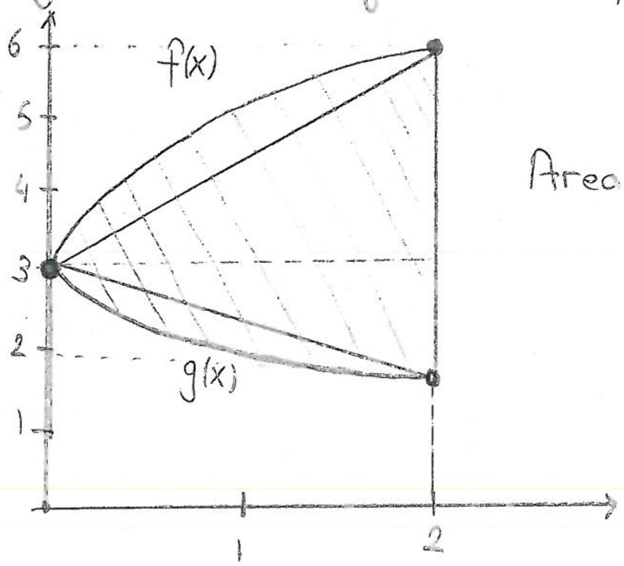
$$= h(\sqrt{x^2})\cdot\frac{1}{2\sqrt{x^2}}\cdot 4x^2 + 2\cdot\int_1^{\sqrt{x^2}} h(u)du - h(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}$$

$$f''(1) = h(1)\cdot\frac{1}{2}\cdot 4\cdot 1 + 2\cdot\int_1^1 h(u)du - h(1)\cdot\frac{1}{2\cdot 1}$$

$$= 4 + 2\cdot 0 - 1 = 3.$$

(b) Let $f(x)$ and $g(x)$ be differentiable functions on $[0,2]$ with the following properties: $2f(0) = f(2) = 2g(0) = 3g(2) = 6$, $f'(x) > 0 > g'(x)$ and $g''(x) > 0 > f''(x)$ for all x in $[0,2]$. Prove that the area of the region between $y = f(x)$ and $y = g(x)$ on $[0,2]$ is greater than 4.

$f(x)$ is increasing, concave down on $[0,2]$.
$g(x)$ is decreasing, concave up on $[0,2]$.



Area between f & g > Area of triangle

$$= \frac{(6-2)\cdot 2}{2} = 4$$