METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY FINAL	
Code : *MAT 119* Acad.Year: *2012-2013* Semester : *SPRING* Date : *04.06.2013* Time : *09:00* Duration : *110 minutes*	Last Name: Name : Student No.: Department: Section: Signature: 6 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1. (10)	2. (35)	3. (20)	4. (15)	5. (5)	6. (15)	Bonus	

Show your work! Please draw a $\boxed{\text{box}}$ around your answers!

1. *(5+5=10pts)* Find the following limits.

(a) $\lim\limits_{x\to 0} \dfrac{\sec x - 1}{x^2} \overset{L'H.R}{=} \lim\limits_{x\to 0} \dfrac{\sec x \tan x}{2x} = \underbrace{\lim\limits_{x\to 0} \dfrac{\sec^2 x}{2}}_{1/2} \cdot \underbrace{\lim\limits_{x\to 0} \dfrac{\sin x}{x}}_{1}$

$= \dfrac{1}{2}$

(b) $\lim\limits_{x\to 1^+} \left(\dfrac{x}{x-1} - \dfrac{1}{\ln x}\right) = \lim\limits_{x\to 1^+} \dfrac{x\ln x - (x-1)}{(x-1)\ln x} \overset{L'H.R}{=} \lim\limits_{x\to 1^+} \dfrac{1.\ln x + \cancel{x.\frac{1}{x}} - \cancel{1}}{1.\ln x + (x-1)\frac{1}{x}}$

$\overset{L'H.R}{=} \lim\limits_{x\to 1^+} \dfrac{\frac{1}{x}}{\frac{1}{x} + \frac{1.x-(x-1).1}{x^2}}$

$= \lim\limits_{x\to 1^+} \dfrac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}}$

$= \dfrac{1}{2}$

2. *(5x7=35pts)* Compute the following integrals.

(a) $\int \sin^3 x \cos^2 x dx = \int -(1-u^2)\,u^2 du = \int (u^4 - u^2)du$

Say, $\cos x = u$

$-\sin x\, dx = du$

$\sin^2 x = 1 - u^2$

$= \dfrac{u^5}{5} - \dfrac{u^3}{3} + C$

$= \dfrac{\cos^5 x}{5} - \dfrac{\cos^3 x}{3} + C$

(b) $\int \dfrac{1}{x^3\sqrt{x^2-1}}dx = \int \dfrac{\sec\theta \cancel{\tan\theta}\, d\theta}{\sec^3\theta \cancel{\tan\theta}} = \int \cos^2\theta\, d\theta$

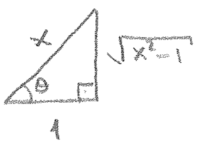
Say, $x = \sec\theta$

$dx = \sec\theta \tan\theta\, d\theta$

$= \int \dfrac{1+\cos 2\theta}{2} d\theta$

$\sqrt{x^2-1} = \tan\theta$

$= \dfrac{\theta}{2} + \dfrac{\sin 2\theta}{4} + C$

$\Rightarrow$ 

$\left.\begin{aligned} \theta &= \tan^{-1}(\sqrt{x^2-1}) \\ \sin\theta &= \frac{\sqrt{x^2-1}}{x} \\ \cos\theta &= \frac{1}{x} \\ \sin 2\theta &= \frac{2\sqrt{x^2-1}}{x^2} \end{aligned}\right\} \Rightarrow \; = \dfrac{\tan^{-1}(\sqrt{x^2-1})}{2} + \dfrac{\sqrt{x^2-1}}{2x^2} + C.$

(c) $\int e^x \sin x dx = e^x(-\cos x) - \int e^x(-\cos x)dx$

Say, $e^x = u \Rightarrow e^x dx = du$

$\sin x\, dx = dv \Rightarrow v = -\cos x$

Say $e^x = \bar{u} \Rightarrow e^x dx = d\bar{u}$

$-\cos x\, dx = d\bar{v} \Rightarrow \bar{v} = -\sin x$

$= e^x(-\cos x) - \left[e^x(-\sin x) - \int e^x(-\sin x)dx\right]$

$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \underbrace{\int e^x \sin x\, dx}$

$2\int e^x \sin x\, dx = e^x(\sin x - \cos x)$

$\Rightarrow \int e^x \sin x\, dx = \dfrac{e^x}{2}(\sin x - \cos x) + C$

(d) $\int \frac{1}{x^3+x}dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+1}\, dx$

$A(x^2+1) + (Bx+C)x = 1$

$(A+B)x^2 + Cx + A = 1$

$\Rightarrow \left.\begin{array}{r} A+B=0 \\ C=0 \\ A=1 \end{array}\right\} \Rightarrow B=-1$

$= \int \frac{1}{x} - \frac{x}{x^2+1}\, dx$

say $x^2+1 = u$

$x\,dx = \frac{du}{2}$

$= \ln|x| - \frac{1}{2}\ln|x^2+1| + C$

$= \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$

(e) $\int_{-2}^{1} \frac{1}{x^5}dx = \int_{-2}^{0} \frac{1}{x^5}dx + \int_{0}^{1} \frac{1}{x^5}dx$

$= \lim_{s\to 0^-} \int_{-2}^{s} \frac{1}{x^5}dx + \lim_{t\to 0^+} \int_{t}^{1} \frac{1}{x^5}dx$

$= \lim_{s\to 0^-} \left.\frac{x^{-4}}{-4}\right|_{-2}^{s} + \lim_{t\to 0^+} \left.\frac{x^{-4}}{-4}\right|_{t}^{1}$

$= \lim_{s\to 0^-} \left(\frac{s^{-4}}{-4} - \frac{(-2)^{-4}}{-4}\right) + \lim_{t\to 0^+} \left(\frac{1}{-4} - \frac{t^{-4}}{-4}\right)$

$= -\infty + \infty$ So, this improper integral is divergent.

3. *(10+10=20pts)* Let R be the region under $y = 3x - x^2$ and above the x-axis. Write the following volumes as integrals. DO NOT COMPUTE THE VALUE OF THE INTEGRALS.

(a) The volume of the solid obtained by rotating R around the line $y = -1$.

$3x - x^2 = 0 \Rightarrow x = 0; x = 3$

$$V = \int_0^3 \pi \left[\left((3x - x^2) - (-1) \right)^2 - \left(0 - (-1) \right)^2 \right] dx$$

(b) The volume of the solid obtained by rotating R around the line $x = -2$.

$$V = \int_0^3 2\pi \left(x - (-2) \right) (3x - x^2) \, dx$$

4. *(7+8=15pts)*

(a) Write the arclength of $y = \sin x$ on $[0, \pi/2]$ as an integral. DO NOT COMPUTE THE VALUE OF THE INTEGRAL.

Arclength formula: $\int_a^b \sqrt{1+(f'(x))^2}\, dx$

$$L = \int_0^{\pi/2} \sqrt{1+\cos^2 x}\, dx$$

(b) Write the surface area of the solid obtained by rotating the region under $y = \sin x$ on $[0, \pi/2]$ around the x-axis as an integral. DO NOT COMPUTE THE VALUE OF THE INTEGRAL.

Surface area of revolution formula: $\int_a^b 2\pi f(x) \sqrt{1+(f'(x))^2}\, dx$

$$S = \int_0^{\pi/2} 2\pi \sin x \sqrt{1+\cos^2 x}\, dx$$

5. *(5pts)* Let $f(x) = (\cos x)^{x^2+1}$. Find $f'(x)$.

$$\ln(f(x)) = \ln(\cos x)^{x^2+1} \Rightarrow \ln(f(x)) = (x^2+1)\ln(\cos x)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = 2x.\ln(\cos x) + (x^2+1).\frac{-\sin x}{\cos x}$$

$$\Rightarrow f'(x) = \left[2x.\ln(\cos x) - (x^2+1)\tan x\right].(\cos x)^{x^2+1}$$

6. *(15pts)* A family of parabolas parametrized by c is given by the following equation $f(x) = (x-c)(x-c^2)$. While c is changing corresponding parabola is also changing. When c=2, and c'=1, how fast is the area between the parabola and x-axis changing?

$(x-c)(x-c^2)=0 \Rightarrow x=c \,;\, x=c^2$

$$\text{Area}(c) = -\int_c^{c^2} (x-c)(x-c^2)\,dx$$

$$\text{Area}(c) = -\int_c^{c^2} x^2 - x(c+c^2) + c^3 \, dx$$

$$= -\left(\frac{x^3}{3} - \frac{x^2}{2}(c+c^2) + c^3 x\right)\Bigg|_c^{c^2}$$

$$= -\left\{\left(\frac{c^6}{3} - \frac{c^4(c+c^2)}{2} + c^5\right) - \left(\frac{c^3}{3} - \frac{c^2(c+c^2)}{2} + c^4\right)\right\}$$

$$= -\left\{\left(\frac{c^5}{2} - \frac{c^6}{6}\right) - \left(\frac{c^4}{2} - \frac{c^3}{6}\right)\right\}$$

$$\Rightarrow \text{Area}'(2) = -\left[\left(\frac{5 \cdot c^4}{2} - \frac{6c^5}{6}\right) - \left(\frac{4c^3}{2} - \frac{3c^2}{6}\right)\right] \cdot c' \Big|_{c=2}$$

$$\text{Area}'(2) = -\left[\left(\frac{5 \cdot 16}{2} - 32\right) - \left(\frac{4 \cdot 8}{2} - \frac{3 \cdot 4}{6}\right)\right] \cdot 1 = +6$$

Bonus. Let $f(x)$ be a continuous, invertible function such that $f(2) = 1, f(5) = 4$ and $\int_2^5 f(x)dx = 10$. Find the value of the integral $\int_1^4 f^{-1}(x)dx$.

$$\int_1^4 f^{-1}(x)\,dx = \int_2^5 s \cdot f'(s)\,ds = \left[s \cdot f(s) - \int f(s)\,ds\right]_2^5$$

Say, $f^{-1}(x) = s$

$x = f(s)$

$dx = f'(s)ds$

$f^{-1}(1) = 2$

$f^{-1}(4) = 5$

$s = u \Rightarrow ds = du$

$f'(s)ds = dv \Rightarrow v = f(s)$

$$= (5 \cdot f(5) - 2 \cdot f(2)) - \int_2^5 f(s)\,ds$$

$$= (5 \cdot 4 - 2 \cdot 1) - 10$$

$$= 8$$