L-functions

Def: $L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is an L-function

EX: If $a_n = 1$ then $L(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is Riemann ζ-function

$L(1) = \infty$
$L(2) = \pi^2/6$
$\vdots$

alt. def.

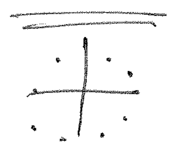
$$L(s) = \prod_{p \text{ prime}} \left(\frac{1}{1-p^{-s}}\right)$$

EX: (Dirichlet L-function)

Def: Dirichlet character is $\chi : (\mathbb{Z}/N\mathbb{Z})^* \longrightarrow \mathbb{C}^*$

w/ $\cdot\ \chi(nm) = \chi(n)\chi(m)$

$\cdot\ \chi(a)^{\phi(N)} = 1$



$\phi(N)^{th}$ roots of unity

Let $\chi(a) = \begin{cases} \chi(a \bmod N) & \gcd(a,N) = 1 \\ 0 & \gcd(a,N) \neq 1 \end{cases}$

$$\Rightarrow L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)\,p^{-s}} \quad \text{(Dirichlet)}$$

$$\Rightarrow \log(\zeta(s)) + \underbrace{\sum_{\substack{\chi \bmod N \\ \chi \neq 1}} \chi(a)^{-1} \log(L(s,\chi))}_{\text{infinite}} = \phi(N)\left(\sum_{p \equiv a \bmod N} \frac{1}{p^s}\right) + g(\ldots$$

ϕ: Euler ϕ-function (finite)

fourier

$$\Rightarrow \#\{\text{primes} \equiv a \bmod N\} = \infty \quad !!!$$

Connection to last time:

EX $S_3(n) = \begin{cases} 0 & n \equiv 7 \bmod 8 \\ \frac{24\sqrt{n}}{\pi} L(1,\chi) & \text{otherwise.} \end{cases}$

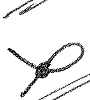
$\rightarrow S_{2n+1}(n)$ always has an answer like this!!

L-functions, Elliptic Curves, and Modular Forms

Given an elliptic curve $y^2 = x^3 + ax + b$ $(*)$

let $a_p = p + 1 - N_p$ ← # {solutions of * mod p}

(*) is $y^2 = x^3$ mod p: "additive reduction".

 mod p: "mult. reduction": split if $(y-\dots)(y+\dots)$, non-split otherwise

$$L_p(T) = \begin{cases} 1 - a_p T + pT^2 & \text{if } (*) \bmod p \text{ nonsingular} \\ 1 - T & \text{if } (*) \text{ split-mult. reduc.} \\ 1 + T & \text{if } (*) \text{ non-split-mult red.} \\ 1 & \text{if } (*) \text{ additive reduct.} \end{cases}$$

$$L(E,s) = \prod_p \frac{1}{L_p(p^{-s})} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

Ex $y^2 + y = x^3 - x^2 - x - 20$ $\qquad \Delta_E = -11^5$

$$L(E,s) = \frac{1}{1 - 11^{-s}} \prod_{\substack{p \neq 11 \\ \text{prime}}} \left(\frac{1}{1 - a_p p^{-s} + p^{1-2s}} \right)$$

$$= 1 - \frac{2}{2^s} - \frac{1}{3^s} + \frac{2}{4^s} + \frac{1}{5^s} + \frac{2}{6^s}$$

Modular form $f(z) = \sum_0^{\infty} a_n q^n$ $\qquad (q = e^{2\pi i z})$

$\Downarrow$?

L-function $L(s,f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$

Conjecture: A series of the form $L(s) = \sum_1^{\infty} \frac{a_n}{n^s}$ w/ $a_n \in \mathbb{Z}$ is the L-function of an elliptic curve (E) w/ conductor N

(Taniyama–Shimura–Weil)

$\Updownarrow$

$L(s)$ is the L-function of a normalized modular form of weight 2 for group $\Gamma_0(N)$

$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \bmod N \right\}$ in $SL_2(\mathbb{Z})$

Birch Swinnerton-Dyer (BSD)

Conjecture: $L(E,s) = C_0 (s-1)^{R_E} + c_1 (s-1)^{R_E+1} + \dots$ ← Taylor series at $s_0 = 1$

w/ $R_E = \text{rank}(E_{\mathbb{Q}})$ ← Mordell-Weil Group.

$\cancel{C_0 = |Ш|}$

and $C_0 = \dfrac{|Ш| \cdot \Omega_E \cdot \text{Reg}(E/\mathbb{Q}) \prod_p c_p}{|E_{Tor}(\mathbb{Q})|^2}$

Ш: Tate-Shafarevich; Ω_E: period of elliptic curve; Reg: regulator of E; c_p: Tamagawa #s

EX: $y^2 = x^3 - 1156x$ elliptic curve

$R_E = 2$

$E_{Tor}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$|Ш| = 1$

$C_0 = 6.385159\dots$