Salih Durhan (lecture 2)

- $K = (K, \Gamma, k, v)$ **Valued fields**

$v: K^* \to \Gamma$

$v(ab) \mapsto v(a) + v(b)$

$v(0) := \infty > \gamma$

$v(a+b) \geq \min(v(a), v(b))$

$\mathcal{O}_v = \{a \in K : v(a) \geq 0\}$

$\mathfrak{M}_v = \{a \in K : v(a) > 0\}$

$k = \mathcal{O}_v / \mathfrak{M}_v$

$\bar{\cdot} : \mathcal{O}_v \to k$

- **Hahn Field:** $k((t^\Gamma))$

$\sum_\gamma a_\gamma t^\gamma$ where $\{\gamma : a_\gamma \neq 0\}$ well-ordered

$v(\sum a_\gamma t^\gamma) = \min\{\gamma : a_\gamma \neq 0\}$

- **Hensel's Lemma:** In $k((t^\Gamma))$ if $f \in \mathcal{O}_v[x]$ and "like Newton's App…"
  - $v\, f(0) > 0$
  - $v\, f'(0) = 0$

  then $\exists\, a$ w/ $f(a) = 0$

Problem: In characteristic $p > 0$ Hensels lemma is not entire story

ex

$f(x) = \cancel{x^p - x - t^{-1}}\ \mathbb{F}_p((t^{\mathbb{Q}}))$

$a = t^{-1/p} + t^{-1/p^2} + \dots$

$f(a) = 0$

$g(a) = 0$ but can't tell this via Newton's approx.

Alt. $g = t f(x) = t x^p - t x - 1$ $g(a) = 0$

but $v(g(0)) = v(-1) = 0$

$v(g'(0)) = v(-t) > 0$

- **Thm** In char K = char $k = 0$,

K Henselian $\Longleftrightarrow$ K is algebraically maximal.

(no proper alg. extension preserves k, v)

~~Thm In pos. characteristic,~~ (characteristic = 0)

Thm: Let $K \leq K(a)$ be an algebraic (valued field) extension. Then there is a sequence $\{a_\rho\} \in K$ w/ $\{a_\rho\} \to a$ and has no pLimit in K. AND if $f(x)$ min polyn. of a/K

Def: · $\{a_\rho\}$ is PC (pseudo-Cauchy) if $v(a_{\rho+1}-a_\rho)$ is eventually increasing.

· $\{a_\rho\} \to a$ if $v(a-a_\rho)$ is eventually increasing

Thm: w/ same setup $K \leq K(a)$ w/ $f(x)$ min polyn. of a in char > 0.

$$f(a_\rho + x) = f(a_\rho) + f_1(a_\rho)x + f_2(a_\rho)x^2 + \dots + f_{p^i}(a_\rho)x^{p^i} + \dots$$

~~Nee~~ (Need terms up to $f_{p^i}(a_\rho)x^{p^i}$ some i + (smaller stuff) (p = characteristic))

Thm: K alg. maximal $\Longleftrightarrow$ $\forall f/K$ $\{v(f(a)) : a \in K\}$ has a maximal elmt

(i.e. every polynomial has a "best" approx. to zero)

Thm: In char $p > 0$

K alg. maximal $\Longleftrightarrow$ $\forall$ $f = c + \underbrace{a_0 x + a_1 x^p + a_2 x^{p^2} + \dots}_{\text{additive polynom.}}$ $\{v(f(a)) : a \in K\}$ has maximal elmt

Def: An additive polyn. in $x_1, \dots, x_k$ is

$$\begin{aligned} & a_{10}x_1 + a_{11}x_1^p + a_{12}x_1^{p^2} + \dots + a_{1k}x_1^{p^k} \\ + \; & a_{20}x_2 + \dots \qquad\qquad + a_{2k}x_2^{p^k} \\ + \; & a_{30}x_3 + \dots \qquad\qquad + a_{3k}x_3^{p^k} \\ + \; & \dots \end{aligned}$$

Thm: Over $\mathbb{F}_q((t))$ every additive polynomial has a "best approximation to 0" ($\{v f(a)\}$ has max elmt)

Thm: If $\Gamma = \mathbb{Z}$ then every polynom has "best approx to 0"

FACT: $Th(\mathbb{F}((t)))$ is UNKNOWN !!!

Question: Which fields admit "best approx" wrt additive polynom?
(Not just $\mathbb{F}_p((t))$)

Question: Additive polynom "best approx" $\Longrightarrow$ All polynom "best approx"?
(in char p answer is "yes")

Def: σ-polynom is $f(x, \sigma(x), \sigma^2(x), \dots, \sigma^n(x)) = F(x)$
w/ $\sigma \in \mathrm{End}(K)$.

- $F(a+x) = F(a) + \sum_i' F_{(i)}(a)\, \sigma^i(x)$

  w/ $i \in \mathbb{N}^{n+1}$, $\sigma^i(x) = x^{i_0} \sigma(x)^{i_1} \dots \sigma(x)^{i_n}$

- Consider a polyn. / K w/ char $K = p > 0$
  - $\to$ replace x^p by $\sigma(x)$
    x^{p^k} by $\sigma^k(x)$
  - $\to$ replace x^n by $x^{i_0} \sigma(x)^{i_1} \dots \sigma^k(x)^{i_k}$
    w/ $n = i_0 + i_1 p + i_2 p^2 + \dots + i_k p^k$

- Let $f(x) \in K[x]$ and $F(x)$ assoc σ polyn.

Def: (F, a) is in σ-Hensel configuration if ~~$\exists\sigma\ \sum\ |i|=1$~~
$\exists\, \gamma$ and i w/ only one $i_k = 1$, others $= 0$
$(|i| = 1)$ (def $|(i_0, \dots, i_n)| = i_0 + \dots + i_n$)

$$v\, F(a) = F_{(i)}(a) + \sigma^i(\gamma) < v\, F_J(a) + \sigma^J \gamma$$

(a linear term is dominant)

Thm: K alg max & k perfect & Γ p-divisible
then all additive polynom have "best approx".

Thm: There are fields w/ best approx for all additive polyn.
BUT not best approx. for all polynom.

→ $\mathbb{F}_p((t^{\mathbb{Q}}))$ is counter-example...