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Quantum Calculus II

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Notation from last time:

V a $\mathbb{C}$-linear space

$M(V) = \bigcup_{n\in\mathbb{N}} M_n(V)$

operations $\quad v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix}$

$a \cdot v \cdot b \in M_{kt}(V)$ for $a \in M_{ks}(\mathbb{C})$ $v \in M_s(V)$ $b \in M_{st}(\mathbb{C})$

$V \xrightarrow{\varphi} W \rightsquigarrow \varphi^{(\infty)} : M(V) \to M(W)$

$\mathcal{B} \subset M(V)$ __abs. matrix convex__ iff $\mathcal{B} \oplus \mathcal{B} \subseteq \mathcal{B}$

$a\mathcal{B} b \subset \mathcal{B}$ all $a, b \in M(\mathbb{C})$

$p = \{\mathcal{B}\}$ filtered base of abs matrix convex sets w/ $\bigcap \mathcal{B} = \{0\}$

locally convex top on $M(V) \Rightarrow p|_V$ locally convex top.

"Quantum Topology"

Quantum topology restricting to a classical topology is "Quantization of topology"

Quantum Duality

Let (V, W) be a $\mathbb{C}$-dual pair w/ ~~duality~~ pairing $\langle \cdot, \cdot \rangle : V \times W \to \mathbb{C}$

Note $W \hookrightarrow V^{\#}$ "algebraic dual" (functionals on V)

→ Want to lift this to $M(W)$ and $M(V)$

→ Scalar pairing $M_n(V) \times M_n(W) \to \mathbb{C}$

$$\langle v, w \rangle = \sum_{i,j} \langle v_{ij}, w_{ij} \rangle$$

(let $b_w = \{ v \in V \text{ w/ } |\langle v, w\rangle| \leq 1 \} = \{w\}^\circ$

$\rightsquigarrow$ weak topology on V $\sigma(V, W)$

Scalar pairing $\rightsquigarrow$ topology

$\sigma(M_n(V), M_n(W)) = \sigma(V, W)^{n^2}$)

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Better duality: $\langle\langle \cdot , \cdot \rangle\rangle : M(V) \times M(W) \to M$

$$\langle\langle v, w \rangle\rangle = [\langle v_{ij}, w_{st} \rangle]_{(i,s),(j,t)}$$

Ex $V = W = \mathbb{C}$ then

$$\langle\langle a, b \rangle\rangle = a \otimes b$$

Def: If $\mathcal{B} \subseteq M(V)$ a quantum set then

$$M(W) \supseteq \mathcal{B}^{\odot} = \{ w \in M(W) \mid \sup \| \langle\langle \mathcal{B}, w \rangle\rangle \| \leq 1 \}$$

(1997) **Thm (Effros-Webster):**

If $\mathcal{B}$ abs. mat. convex then $(\mathcal{B}^{\odot})^{\odot} = \mathcal{B}^{-}$ weak closure.

Min and Max Quantisizations

Let $b \subseteq V$ be an abs. convex set $\left(\begin{matrix} b_n = 0 & n > 1 \\ b_1 = b & \end{matrix} \right)$

$$b_{max} = (b^{\odot})^{\odot}$$

$$b_{min} = (b^{\circ})^{\odot}$$

Cor of E-W: If $\mathcal{B}$ is abs convex matrix set w/ $b = b_1$ then

$$b_{max} \subseteq \mathcal{B} \subseteq b_{min}$$

- Let $t = \{b\}$ be a topology (filtered base) in V compatible w/ duality (i.e. $(V, t)' = W$)

write

$$\max(t) = \{ b_{max} \mid b \in t \}$$

$$\min(t) = \{ b_{min} \mid b \in t \}$$

Quantum Topologies

Cor: Then any quantum topology p (of $M(V)$) w/ $p|_V = t$ satisfies $\min(t) \leq p \leq \max(t)$

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Thm (Dosi - 2011): Let (V, W) be a dual pair. Then $\min \sigma(V,W) = \max \sigma(V,$

$\Rightarrow$ "Weak topology has only one quantization! Weak quantum topology."