İbrahim Ünal

ϕ-Free Embeddings

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Setup:

(M^n, g) is a Riemannian manifold

ϕ a p-form on M is a "calibration" if

① $d(\phi) = 0$ (closed)

② comass$(\phi) \leq 1$

For any p-plane $\vec{\xi}$ in any tangent sp.
($\vec{\xi} = e_1 \wedge \cdots \wedge e_p$ where $\{e_1, \ldots, e_p\}$ is oriented orthonormal basis)

$\phi(\vec{\xi}) = \phi(e_1, \ldots, e_p) \leq 1$

For non-orthon. bases, compare to $\mathrm{Vol}(e_1, \ldots, e_p)$

Notation: If $\phi(\vec{\xi}) = 1$ then we call $\vec{\xi}$ a "ϕ-plane" or "calibrated plane"

$$G(\phi) = \{\text{all } \phi\text{-planes}\}$$

("Contact set of ϕ")

Def: A p-dim'l submanifold $N^p \subset M^n$ is "calibrated" if all of its tangent spaces are ϕ-planes.

This means that ϕ restricts to N to give Volume form

Fundamental "Lemma" of Calibrations

Thm: Every calibrated submanifold is volume minimizing in its homology class.

Proof:

Let $[N] = [N'] \in H_p(M^n)$.

$$\mathrm{Vol}(N) = \int_N \phi = \int_{N'} \phi \leq \mathrm{Vol}(N') \quad \blacksquare$$

If N is calibrated

Stokes' Thm

Maybe N' not calibrated

Note: $\phi : Gr_p(M^n) \longrightarrow [-1, 1]$

EX: $(\mathbb{R}^2, dx)$

1) $d(dx) = 0$
2) $dx\left(a\,\partial/\partial x + b\,\partial/\partial y\right) = a \le 1$ — $a^2+b^2=1$

$G(dx) = \{\partial/\partial x\}$ "Contact Set"

Calibrated Submanifolds are Horizontal Lines

Note: Lines are geodesics w.r.t. flat metric $\rightsquigarrow$ minimizing!

EX: If M^{2n} is a Kähler mfld then ω (Kähler form) is a calibration!

$\to$ Recall $\omega(X,Y) = g(JX, Y) \le \|JX\|\cdot\|Y\| = \|X\|\cdot\|Y\|$

Equality when $JX \parallel Y$

$\Rightarrow$ Calibrated 2-planes ("ω-planes") are $\mathbb{C}$-lines!!

More generally Calibrated submanifolds are $\mathbb{C}$-curves.

Generalization: $\left(M^{2n}, \frac{\omega^p}{p!}\right)$ is also a calibration.

Calibrated submanifolds are p-dim'l $\mathbb{C}$-submflds.

EX: M^{4n} Hyper-Kähler Manifold (uses I, J, K Quaternions)

$\Phi = \frac{1}{6}\left(\frac{\omega_I^2}{2} + \frac{\omega_J^2}{2} + \frac{\omega_K^2}{2}\right)$ has $G(\Phi) = \{\mathbb{H}\text{-lines}\}$

calibrates I $\mathbb{C}$-planes; calibrates K $\mathbb{C}$-planes

Def: A submanifold $N^k \subset M^n$ is "ϕ-free" if no tangent spaces (at all points) contains a ϕ-plane.

Note: If $k<p$, this is always true
$k=p$ this is probably true
$k>p$ this is more tricky.

EX: ϕ-free in Kähler mfld means tangent space has no $\mathbb{C}$-lines ("totally real")

Next Time: Existence of ϕ-free embeddings

Also some more standard calibrations.

Q: Which manifolds can be embedded in M^n as ϕ-free?

EX: $(\mathbb{H}^n,\ \Omega = \tfrac{1}{3}(\tfrac{\omega_1^2}{2} + \tfrac{\omega_2^2}{2} + \tfrac{\omega_3^2}{2}))$ $\rightsquigarrow$ "Ω-free" means no quaternion lines

- free dimension $(\Omega) = 3n$

Let $N^m \hookrightarrow \mathbb{H}^n$

$\hookrightarrow$ If $n \le 3m$ can you find an isotopy so that the resulting map is Ω-free?

$\boxed{\textbf{Thm (Unal):}\ \text{If } 5m < 12n+4 \text{ then YES.}}$

EX: $(\mathbb{R}^7,\ \phi = \text{associative})$

free dimension $= 4$

$\boxed{\textbf{Thm (classical):}\ \text{Every 3-dim'l manifold can be embedded } \phi\text{-free}}$

EX: $(\mathbb{R}^7,\ *\phi = \text{coassociative})$

free dimension $= 4$

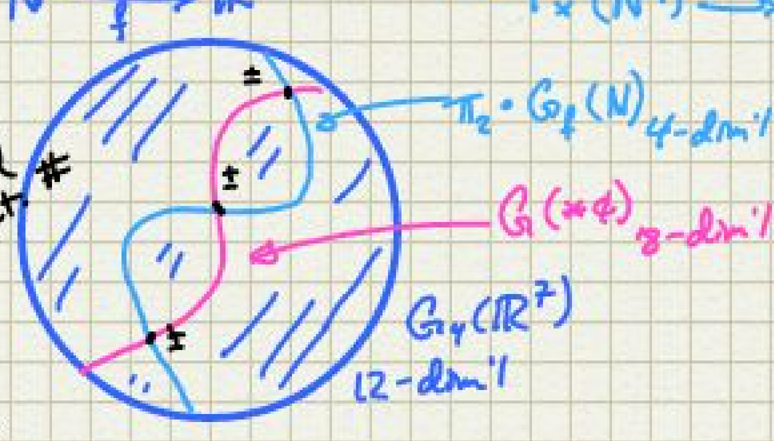
$\boxed{\textbf{Thm (Unal):}\ \text{If } N^4 \hookrightarrow (M^7, \phi) \text{ is } *\phi\text{-free embedding into } G_2\text{-mfld then } \chi(N) = 0\ !}$

Note: From Donaldson's Thm every 4-dim'l mfld embeds into $\mathbb{R}^7$

Consider Grassman bundle on embedding

$$\begin{tikzcd} (\mathbb{R}^7 \times G(*\phi))^c \arrow[r, hook] & \mathbb{R}^7 \times G_4(\mathbb{R}^7) \arrow[r, "\pi_2"] \arrow[d] & G_4(\mathbb{R}^7) \\ N^4 \arrow[ur, dashed, "G_f"] \arrow[r, hook, "f"'] & \mathbb{R}^7 & \end{tikzcd}$$

G_f is Gauss map induced by $T_x(N^4) \longrightarrow T_{f(x)}(\mathbb{R}^7)$



Note: Intersections are transverse. "Whitney trick" says they can be separated if algebraic intersect # is 0

$\Downarrow$

$\chi(N) = 0$

Def: Let $A \subset \tilde{G}_p(M^n)$ be a subset of the Grassmannian manifold of M. Then $N^p \overset{f}{\hookrightarrow} M^n$ is ***"A-directed"*** if $G_f(N) \subset A$.

Thm (Eliashberg–Gromov) Suppose A is an open subset and the corresp. open differential relation $R_A \subset Jet^1(N,M)$ ~ 1-Jets: (points & first derivatives) is ample, then every embedding

$$f_0 : N \hookrightarrow M$$

whose Gauss map

$$G_{f_0} : N \longrightarrow G_p(M^n)$$

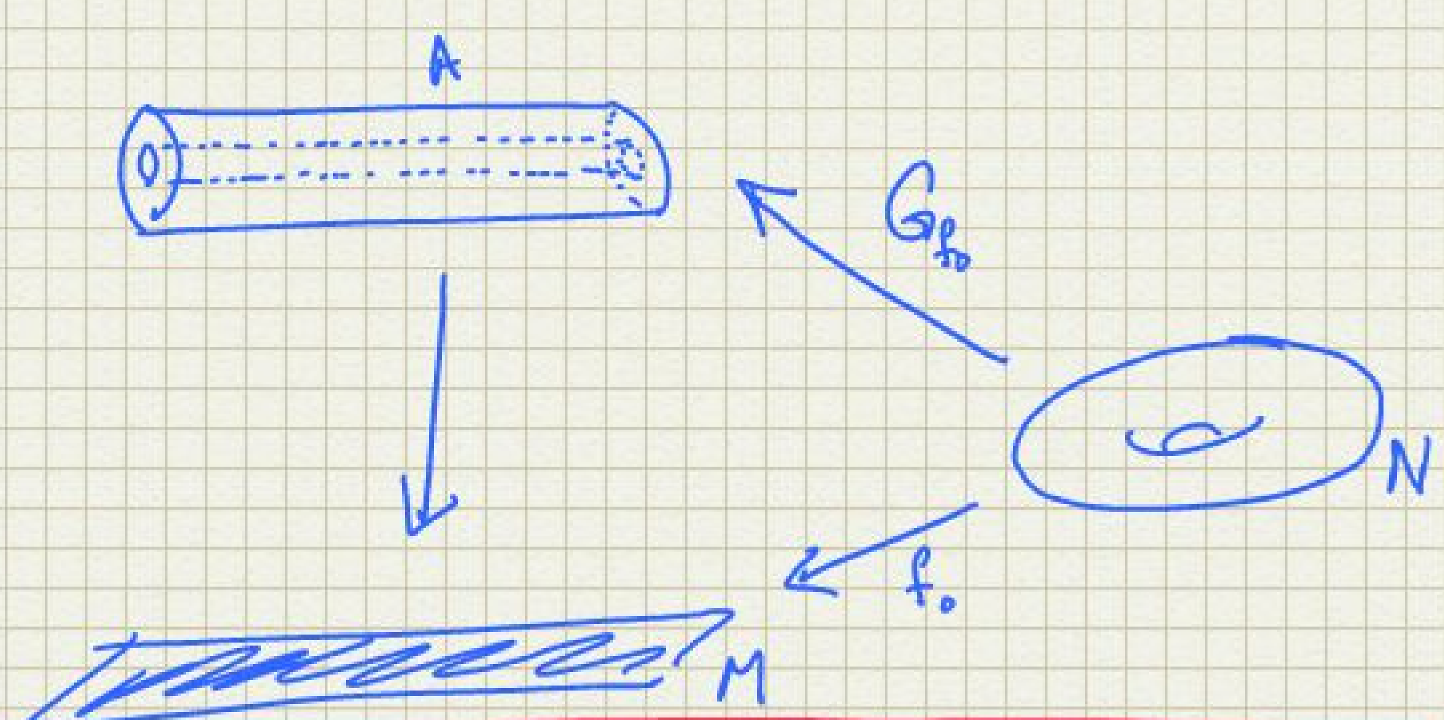
is homotopic over N to a map

$$G : N \longrightarrow A \subset G_p(M^n)$$

can be isotoped to an A-directed embedding.

(Ample means "everything is in the convex hull")

Note: If D is a linear subsp and D^c is ample, then $\mathrm{codim}(D^c) \geq 2$



Thm (Eliashberg–Gromov) R_A is ample iff for every $x \in M$ and $S \in G_{p-1}(A_x)$ the set

$$\Omega_S = \{ v \in T_x M \text{ with } \mathrm{Span}(S, v) \in A_x \}$$

is ample.