Özgür Kişisel — Quantum Groups — 14/5/2013

Recall: $(V, W, b, b', d, d', c, c^-)$ is representation data for tangle T if relations from before are satisfied.

$$\begin{bmatrix} F(\cup) = b : k \longrightarrow V\otimes W \\ F(\overleftarrow{\cup}) = b' : k \longrightarrow W\otimes V \end{bmatrix} \qquad \begin{bmatrix} F(X^+) = c : V\otimes V \longrightarrow V\otimes V \\ F(X^-) = c^- : V\otimes V \longrightarrow V\otimes V \end{bmatrix}$$

$$\begin{bmatrix} F(\cap) = d : W\otimes V \longrightarrow k \\ F(\overleftarrow{\cap}) = d' : V\otimes W \longrightarrow k \end{bmatrix}$$

Def: $\left.\begin{matrix} c : V\otimes V \longrightarrow V\otimes V \\ \mu : V \longrightarrow V \end{matrix}\right\}$ (c, μ) is an enhanced R-matrix if

(1) c satisfies Yang-Baxter Eqn
$$(c\otimes id)(id\otimes c)(c\otimes id) = (id\otimes c)(c\otimes id)\ldots$$

(2) $c(\mu\otimes\mu) = (\mu\otimes\mu)c$

(3) $tr_2\left(c^{\pm 1} \circ (id_V \otimes \mu)\right) = id_V$

(4) $\left(\tau c^{\mp 1}\right)^+ \left(id_{V^*}\otimes\mu\right)\left(c^{\pm 1}\tau\right)^+\left(id_{V^*}\otimes\mu^{-1}\right)$
$$\shortparallel$$
$$id_{V^*\otimes V}$$

- c = R-matrix
- μ = "correction map" ("normalization factor")

- Def of tr_2:

If $\{v_i\}$ basis for V, then $\{v_i\otimes v_j\}$ basis for $V\otimes V$.

Given endomorphism $A : V\otimes V \longrightarrow V\otimes V$

have $$A(v_i\otimes v_j) = \sum_{k,l} A_{ij}^{kl}\, v_k\otimes v_l$$

$tr_2(A) : V \longrightarrow V$ by $$tr_2(A)(v_i) = \sum_{k,l} A_{il}^{kl}\, v_k$$

- $\tau : V\otimes V \longrightarrow V\otimes V$ is twist map $\left(\tau(a\otimes b) = b\otimes a.\right)$

• $(-)^{+}$ and $(-)^{\times}$ are "partial duals"

$$A : V \otimes W \longrightarrow U \otimes Z \quad \text{by} \quad A(v_i \otimes w_j) = A_{ij}^{k\ell}\, u_k \otimes w_\ell$$

(Recall: <u>Ordinary duals</u>:

$$T : V \to W \quad \text{by} \quad T v_i = T_i^j w_j \qquad \begin{pmatrix} v^i = v_i^* \\ w^i = w_i^* \end{pmatrix}$$

$$T^* : W^* \to V^* \quad \text{by} \quad T^* w^i = T_j^i v^i \qquad \text{transpose } i \,\&\, j$$
)

$\to$ $A^{+} : U^* \otimes W \longrightarrow V^* \otimes Z$ by

$$A^{+}(u^i \otimes w_j) = A_{jk}^{i\ell}\, v^k \otimes z_\ell \qquad \text{transpose } i \,\&\, k$$

$\to$ $A^{\times} : V \otimes Z^* \longrightarrow U \otimes W^*$ by

$$A^{\times}(v_i \otimes z^j) = A_{i\ell}^{kj}\, u_k \otimes w^\ell \qquad \text{transpose } j \,\&\, \ell$$

<u>Thm</u>: Given an enhanced R-matrix (c, μ) on a f.d. v.s. V, there is a unique strict tensor functor

$$F : \mathcal{T} \to \mathcal{V}$$

so that

$$F(+) = V, \quad F(-) = V^*, \qquad F(X_+) = c, \qquad F(\cup) = \delta_V \ \text{(coevaluation)}, \quad F(\overleftarrow{\cup}) = (id_{V^*} \otimes \mu^{-1})\,\delta_{V^*}$$

↳ This forces

$$F(\cap) = ev_V, \quad F(\overleftarrow{\cap}) = ev_{V^*}(\mu \otimes id_{V^*}), \qquad F(X_-) = c^{-1}.$$

• "<u>coevaluation</u>" is $\delta_V : k \longrightarrow V \otimes V^*$

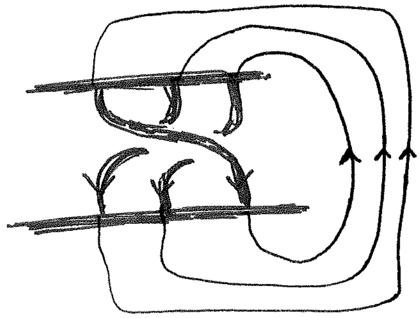
by $\delta_V(1) = \sum v_i \otimes v^i$

• "<u>evaluation</u>" is $ev_V : V \otimes V^* \longrightarrow k$

by $ev_V(v^j \otimes v_i) = v^j(v_i) = \delta_i^j$

Before proving the Thm, we will look at how it is used:

(Back to braids)



Recall:

closure of a braid

Let $\sigma \in B_n$ be an n-strand braid.

$\hat{\sigma}$ = closure of σ (a link)

Given $F : \mathcal{T} \longrightarrow \mathcal{V}$ like in thm,

$$\boxed{\begin{aligned} &\text{Cor: } F(\sigma) = \rho_n^c(\sigma) \\ &\phantom{\text{Cor: }} F(\hat{\sigma}) = tr\left(\mu^{\otimes n} \circ \rho_n^c(\sigma)\right) \end{aligned}}$$

Link invariant.

where ρ_n^c is $\underset{\sigma_i}{| | \cdots \times \cdots | | |} \longmapsto id_V \otimes \cdots \otimes c \otimes \cdots \otimes id_V$

in pos i

Proof of 2^{nd} part of corollary:

$$F(\hat{\sigma}) = \left(ev_{V^* \otimes n} \circ (\mu^{\otimes n} \otimes id_{V^* \otimes n})\right) \circ \left(F(\sigma) \otimes id_{V^* \otimes n}\right) \circ \delta_{V \otimes n}$$

$$= ev_{V^* \otimes n} \circ \left(\mu^{\otimes n} \circ F(\sigma) \circ id_{V^* \otimes n}\right) \circ \delta_{V \otimes n}$$

$$= tr\left(\mu^{\otimes n} \circ F(\sigma)\right)$$