Özgür Kişisel Quantum Groups 7.5.2013

Representations of the category of tangles.

Recall: If $\underline{C}$ is a tensor category then $\underline{C}^{str}$ is equivalent strict tensor cat.

$\Rightarrow$ Let $\underline{\mathcal{V}}$ be the strict tensor category assoc. to $Vect_{finite}(k)$

> **Def:** A representation of the tangle category T is a functor
>
> $$F : \underline{T} \longrightarrow \underline{\mathcal{V}} \qquad \text{(strict tensor functor)}$$

Note: $F : \underline{T} \longrightarrow \underline{\mathcal{V}}$ produces an isotopy invariant of links (k-valued)

$$\underline{\text{Links}} = Mor(\emptyset, \emptyset) \xrightarrow{F} End(F(\emptyset)) = End(k) \cong \underline{k}$$

Note: Value of F on objects of $\mathcal{V}$ is determined by

$$\left.\begin{array}{l} F(+) = V \\ F(-) = W. \end{array}\right\} \quad \underline{\text{ex}}\ F(+-++) = V \otimes W \otimes V \otimes V$$

On morphisms:

$$\left.\begin{array}{lll} F(\downarrow) & F\left(\overset{+}{\underset{+}{\downarrow}}\right) = id_V \\ F(\uparrow) & F\left(\overset{-}{\underset{-}{\uparrow}}\right) = id_W \end{array}\right\} \quad F(\downarrow\uparrow\uparrow) = id_{V\otimes W \otimes W}$$

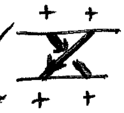
Other generators:

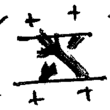
$F(\cup)$ $\quad F\left(\overset{+\ \ -}{\cup}\right) : k \longrightarrow V \otimes W$ call it $\underline{b} : k \to V \otimes W$

$F(\overleftarrow{\cup})$ $\quad F\left(\overset{-\ \ +}{\cup}\right) : k \longrightarrow W \otimes V$ call it $\underline{b'} : k \longrightarrow W \otimes V$

$F(\cap)$ $F($  $): W\otimes V \to k$ call it $\underline{\underline{d}}: W\otimes V \to k$

$F(\overset{\leftarrow}{\cap})$ $F($  $): V\otimes W \to k$ call it $\underline{\underline{d'}}: V\otimes W \to k$

$F(X_+)$ $F($  $): V\otimes V \to V\otimes V$ call it $\underline{\underline{c^+}}: V\otimes V \to V\otimes V$

$F(X_-)$ $F($  $): V\otimes V \to V\otimes V$ call it $\underline{\underline{c^-}}: V\otimes V \to V\otimes V$

→ What do the relations ① - ⑧ look like w/ b, c, d ??

(See notes from last time for pictures:)

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End(V) ① $(id_V \otimes d)(b \otimes id_V) = id_V = (d' \otimes id_V)(id_V \otimes b')$

End(W) ② → same but w/ W and '

End(V⊗V), **End(W⊗W)** ③ $(\underline{d} \otimes id_W \otimes id_W)(id_W \otimes \underline{d} \otimes id_V \otimes id_W \otimes id_W)(id_W \otimes id_W \otimes \ldots$

$= \ldots$

better notation.

$(\underline{d} \otimes id_{W\otimes V})(id_W \otimes \underline{d} \otimes id_{V\otimes W\otimes W\otimes V})(id_{W\otimes W} \otimes c^{\pm} \otimes id_{W\otimes V}) \ldots$

$= \ldots$

(bleah... you can write this yourself...)

End(V⊗V) ④ $c^+ c^- = id_{V\otimes V} = c^- c^+$ $\boxed{c^- = (c^+)^{-1}}$

End(V⊗V⊗V) ⑤ $(c^+ \otimes id_V)(id_V \otimes c^+)(c^+ \otimes id_V) = (id_V \otimes c^+)(c^+ \otimes id_V)(id_V \otimes c^+)$ } Main difficult constraint.

Yang-Baxter Equation

End(V) ⑥ $(id_V \otimes d')(c^{\pm} \otimes id_W)(id_V \otimes b) = id_V$

End(V⊗W) ⑦ $gh = id_{V\otimes W}$ where $\begin{cases} g = (d \otimes id_{V\otimes W})(id_W \otimes c^{\pm} \otimes id_W)(id_{W\otimes V} \otimes b) \\ h = (id_{W\otimes V} \otimes d')(id_W \otimes c^{\mp} \otimes id_W)(b' \otimes id_{V\otimes W}) \end{cases}$

End(W⊗V) ⑧ → same as ⑦ but swapping V & W and g, h

<u>Def:</u> $(V, W; b, b', d, d', c^+, c^-)$ satisfying these requirements is "<u>representation data</u>" for T

<u>Plan:</u> Set $W = V^*$

→ This makes many of these obvious maps

→ Relation 5 turns into Yang-Baxter Eqn.

Only a few choices remain to have representations!