Özgür Kişisel — Quantum Groups — 30.4.2013

Today: Category of Tangles!

T = "category of tangles"

$Ob(T) = \{$ finite sequences of $\pm$ signs $\}$ including empty seq.

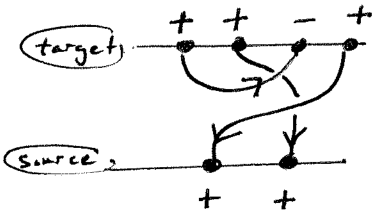
$Mor(T) = \{$ isotopy classes of oriented tangles (preserve sign) $\}$

$f: A \to B$

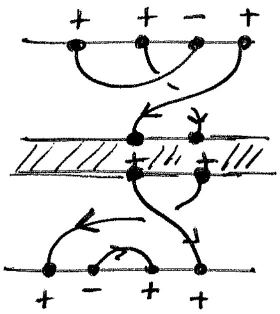
\+ in ~~source~~ A $\rightsquigarrow$ incoming

\+ in ~~target~~ B $\rightsquigarrow$ outgoing

EX

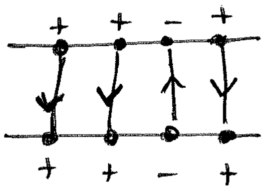


EX



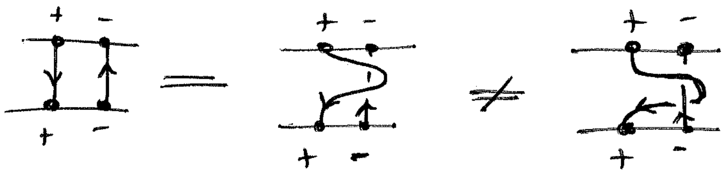
[Composition is glueing w/ matching signs.]

EX



[Identity is "straight arrows" w/ orientation from sign]

Remark Isotopy classes so e.g.

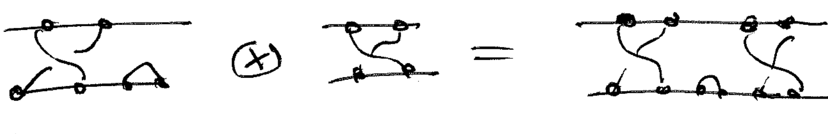


Tangles is a strict tensor category:

- $(+,-,+\cdots) \otimes (-,+,+,\cdots) = (+,-,+,\ldots,-,+,+,\ldots)$

  $(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n) \otimes (\varepsilon_{n+1},\varepsilon_{n+2},\ldots,\varepsilon_m) = (\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n,\varepsilon_{n+1},\varepsilon_{n+2},\ldots,\varepsilon_m)$

- Similarly for morphisms:

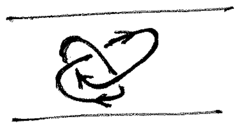


- Unit object is empty tangle

$$A \otimes \emptyset = A$$

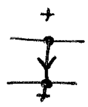
(identity morphism of $\emptyset$ is "empty morphism")

$\rightsquigarrow$ Note $\mathrm{End}(\emptyset) = \{\text{Links}\}$

(EX  is a morphism $\emptyset \to \emptyset$)

Generators and Relations

Notation:

- $\downarrow$ will be  : identity morphism on $(+)$
- $\uparrow$ will be  : identity morphism on $(-)$

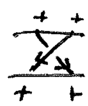
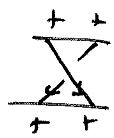
cup

- $\cup$ will be  : morphism $\emptyset \longrightarrow (+,-)$
- $\overleftarrow{\cup}$ will be  : morphism $\emptyset \longrightarrow (-,+)$

cap

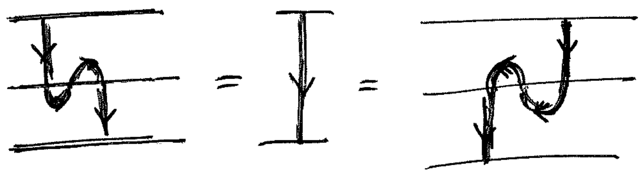
- $\cap$ will be  : morphism $(-,+) \longrightarrow \emptyset$
- $\overleftarrow{\cap}$ will be  : morphism $(+,-) \longrightarrow \emptyset$

cross

- X_+ will be  : morphism $(+,+) \longrightarrow (+,+)$
- X_- will be  : morphism $(+,+) \longrightarrow (+,+)$

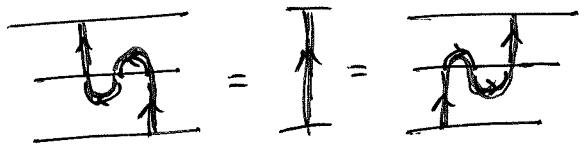
Thm: These generate all tangles (via composition w/ id & $\otimes$)
Furthermore relations are generated by the following
8 relations:

1. $(\downarrow \cap)\circ(\cup \downarrow) = \downarrow = (\overleftarrow{\cap}\downarrow)\circ(\downarrow \overleftarrow{\cup})$



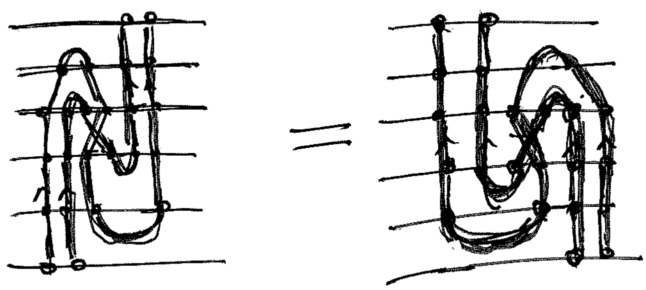
Convention: "$\downarrow \cap$" $= \downarrow \otimes \cap$

2. $(\uparrow \overleftarrow{\cap})\circ(\overleftarrow{\cup}\uparrow) = \uparrow = (\cap \uparrow)\circ(\uparrow \cup)$



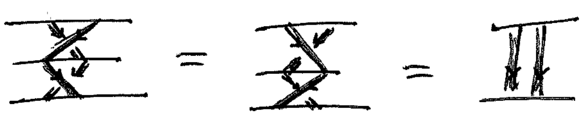
3. $(\cap \uparrow\uparrow)\circ(\uparrow \cap \downarrow\uparrow\uparrow)\circ(\uparrow\uparrow X_\pm \uparrow\uparrow)\circ(\uparrow\uparrow\downarrow \cup \uparrow)\circ(\uparrow\uparrow \cup)$

$= (\uparrow\uparrow \overleftarrow{\cap})\circ(\uparrow\uparrow\downarrow \overleftarrow{\cap}\uparrow)\circ(\uparrow\uparrow X_\pm \uparrow\uparrow)\circ(\uparrow \overleftarrow{\cup}\downarrow\uparrow\uparrow)\circ(\overleftarrow{\cup}\uparrow\uparrow)$

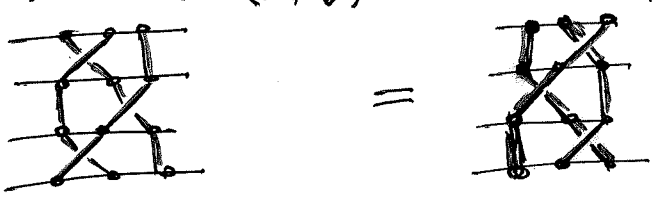


$X_\pm$ with arrows reversed !!

4. $X_+ \circ X_- = X_- \circ X_+ = \downarrow\downarrow$

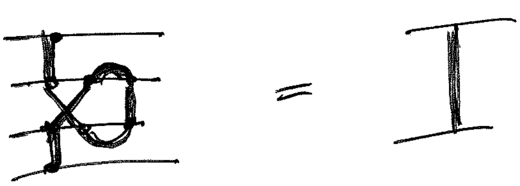


5. $(X_+ \downarrow)\circ(\downarrow X_+)\circ(X_+ \downarrow) = (\downarrow X_+)\circ(X_+ \downarrow)\circ(\downarrow X_+)$



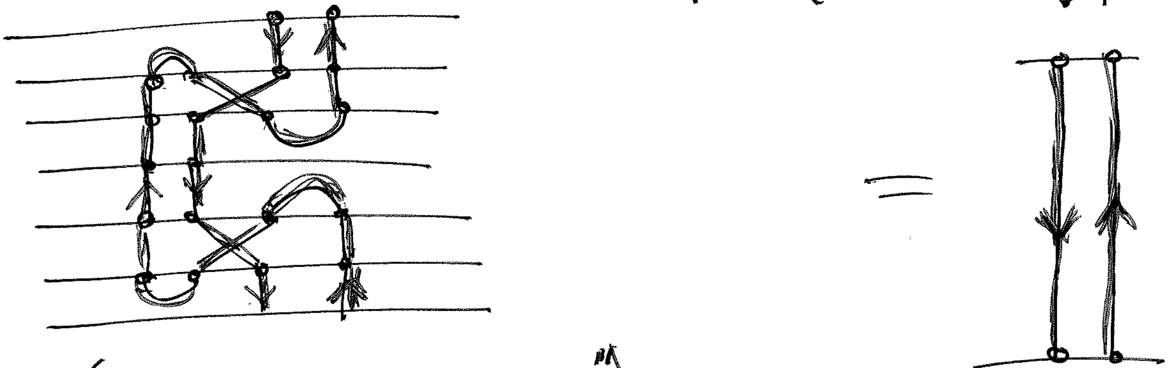
Reidemeister move 3

6. $(\downarrow \overleftarrow{\cap})\circ(X_\pm \uparrow)\circ(\downarrow \cup) = \downarrow$



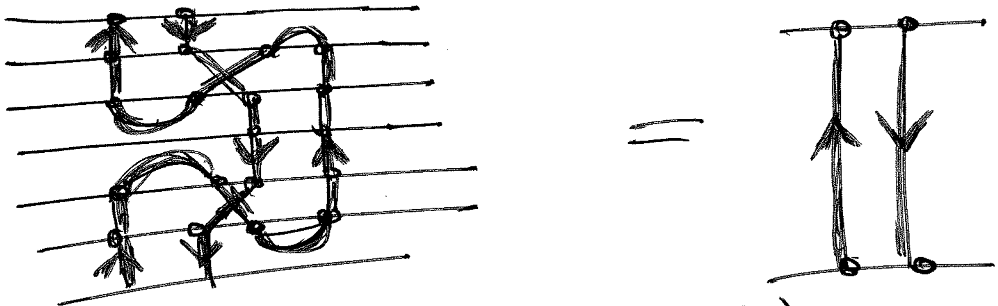
Reidemeister move 2

7) $(\cap\downarrow\uparrow)\circ(\uparrow X_{\pm}\uparrow)\circ(\uparrow\downarrow\cup)\circ(\uparrow\downarrow\overset{\leftarrow}{\cap})\circ(\uparrow X_{\mp}\uparrow)\circ(\overset{\leftarrow}{\cup}\downarrow\uparrow) = \downarrow\uparrow$



("combine #3 and #4")

8) $(\uparrow\downarrow\overset{\leftarrow}{\cap})\circ(\uparrow X_{\pm}\uparrow)\circ(\overset{\leftarrow}{\cup}\downarrow\uparrow)\circ(\cap\downarrow\uparrow)\circ(\uparrow X_{\mp}\uparrow)\circ(\uparrow\downarrow\cup) = \uparrow\downarrow$



(reverse arrows in 7)

Q: Why ~~not~~ include an inversion

$$S : T \to T$$

by reversing sign on objects
arrow on morphisms

Call this a "tensored Hopf category"

$\rightsquigarrow$ modulo S there are less messy relations & generators...?

ex: 8 is S 7

ex: 3 shows how to make $S X_{\pm}$