We were doing category theory.

Previously: Adjoints are equivalent to bijection between Homs.

EX X = a set

$k\{X\}$ = free k-alg. on X

$\mathcal{C}$ = k-algebras
$\mathcal{D}$ = sets

$\Rightarrow$ free & forgetful functors are adjoint

G : sets $\to$ k-alg F : k-alg $\to$ sets
$X \mapsto k\{X\}$ $A \mapsto A$ (forget alg str)

Then G : sets $\rightleftarrows$ k-alg : F adjoint

$$\mathrm{Hom}(k\{X\}, A) \longleftrightarrow \mathrm{Hom}(X, F(A))$$

(algebra maps from free algebras) $\longleftrightarrow$ (maps from generators.)

EX (Tensor products)

$\mathcal{C} = \mathcal{D}$ vector spaces. V = fixed v.s.

$$\left\{ \begin{array}{l} F : \mathcal{C} \to \mathcal{D} \\ U \mapsto \mathrm{Hom}(V, U) \end{array} \right\} \left\{ \begin{array}{l} G : \mathcal{D} \to \mathcal{C} \\ W \mapsto W \otimes V \end{array} \right\}$$

$- \otimes V$: v.sp. $\rightleftarrows$ v.sp : $\mathrm{Hom}(V, -)$ adjoint

$$\mathrm{Hom}(W \otimes V, U) \Longleftrightarrow \mathrm{Hom}(W, \mathrm{Hom}(V, U))$$

Next: Tensor Categories!!

Tensor Categories

Ingredients: ① $\mathcal{C}$ a category

② $-\otimes- : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ a functor $\begin{cases} V, W \rightsquigarrow V \otimes W \\ f, g \rightsquigarrow f \otimes g \end{cases}$

Note: $\begin{cases} (f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g \\ id_{c \otimes d} = id_c \otimes id_d . \end{cases}$

(Prototype: $\mathcal{C}$ = Vector spaces $/k$ w/ usual tensor product)

③ (Associativity constraint)

$\exists$ natural isom

$$a : ((- \otimes -) \otimes -) \Longrightarrow (- \otimes (- \otimes -))$$

(i.e. $a : \otimes(\otimes \times id) \longrightarrow \otimes(id \times \otimes)$)

(Equivalently must have the following commute:

$$\begin{tikzcd} (U \otimes V) \otimes W \arrow[r, "a"] \arrow[d, "(f \otimes g) \otimes h"] & U \otimes (V \otimes W) \arrow[d, "f \otimes (g \otimes h)"] \\ (U' \otimes V') \otimes W' \arrow[r, "a"] & U' \otimes (V' \otimes W') \end{tikzcd}$$
)

④ (Pentagon axiom)

The following pentagon commutes:

$$\begin{tikzcd} & ((U \otimes V) \otimes W) \otimes X \arrow[dl] \arrow[dr] & \\ (U \otimes (V \otimes W)) \otimes X \arrow[d] & & (U \otimes V) \otimes (W \otimes X) \arrow[d] \\ U \otimes ((V \otimes W) \otimes X) \arrow[rr] & & U \otimes (V \otimes (W \otimes X)) \end{tikzcd}$$

⑤ (left/right unit constraint with respect to ~~obj~~ object I)

$\exists$ natural isom

$$l : \otimes(I \times id) \to id \qquad \begin{tikzcd} I\otimes V \arrow[r] \arrow[d] & V \arrow[d] \\ I\otimes V' \arrow[r] & V' \end{tikzcd}$$
$$I \otimes id \to id$$

$$r : \otimes(id \times I) \to id \qquad \begin{tikzcd} V\otimes I \arrow[r] \arrow[d] & V \arrow[d] \\ V'\otimes I \arrow[r] & V' \end{tikzcd}$$
$$id \otimes I \to id$$

(Idea: I is left/right unit)

⑥ (Triangle axiom) (left/right unit structures should be compatible

$$\begin{tikzcd} (V\otimes I)\otimes W \arrow[rr] \arrow[dr] & & V\otimes(I\otimes W) \arrow[dl] \\ & V\otimes W & \end{tikzcd}$$

commutes.

Def: A tensor category is $(\underset{\text{cat.}}{C}, \underset{\text{prod.}}{\otimes}, \underset{\text{unit}}{I}, \underset{\text{assoc.}}{a}, \underset{\text{unit str}}{l, r})$ as described above.

Basic Propositions

- Properties of Unit.

Lemma: The following commutes:

$$\begin{tikzcd} (I\otimes V)\otimes W \arrow[rr] \arrow[dr] & & I\otimes(V\otimes W) \arrow[dl] \\ & V\otimes W & \end{tikzcd}$$

$$\begin{tikzcd} (V\otimes W)\otimes I \arrow[rr] \arrow[dr] & & V\otimes(W\otimes I) \arrow[dl] \\ & V\otimes W & \end{tikzcd}$$

Proof: (Sketch)

Combine pentagon & triangle axioms.
(throw in an extra $I\otimes \cdots$)
→ use isom l at the ~~end. to~~ to get rid of extra I.

∎

Goal: In any tensor category End(I) is a commutative monoid.

(Need one more lemma first)

then examples...

then braided categories...