Symplectic Manifolds

- (M^{2n}, ω) is symplectic if ω is a closed, nondegen. 2-form

  closed $\to d\omega = 0$, nondegen. $\to \underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{n \text{ times}} > 0$

- J is a compatible almost complex ~~symplectic~~ structure if $\omega(\cdot, J\cdot)$ is a Riemannian metric on M.

EX $\mathbb{C}P^2 \# \mathbb{C}P^2$ has no almost complex structure and so is ~~neither~~ not symplectic

$\quad\|$

$\#_2 \mathbb{C}P^2$

Note: $H_2(\#_2 \mathbb{C}P^2)$ nontrivial $(\cong \mathbb{Z} \oplus \mathbb{Z})$ so there are 2-forms

However there is no almost complex structure → see thm below

Thm (Wu): If (X^4, J) is almost complex, then there is $\alpha \in H^2(X, \mathbb{Z})$ w/ $\alpha \equiv w_2$ (2^{nd} Stieffel-Whitney class)

$$c_1(X, J) = \alpha$$

$$\alpha \cup \alpha = 2e(X) + 3\sigma(X).$$

(e: Euler char, σ: signature)

Furthermore, if such an α exists, then there is an almost $\mathbb{C}$-str. on X

In the example, $H^2 = \mathbb{Z} \oplus \mathbb{Z}$ so $\alpha \in H^2$ is $\alpha = ah_1 + bh_2$

$$\left.\begin{array}{l} e(\#_2 \mathbb{C}P^2) = 4 \\ \sigma(\#_2 \mathbb{C}P^2) = 2 \end{array}\right\} \alpha^2 = 2 \cdot 4 + 3 \cdot 2 = 14$$

but $14 \neq (ah_1 + bh_2)^2 = a^2 + b^2$ for int. a, b

Thm An oriented smooth mfld X^4 has an almost $\mathbb{C}$-str $\iff$ $b_1 - b_2^+$ is odd.

EX: $\#_3 \mathbb{CP}^2$ is not symplectic but has an almost $\mathbb{C}$-str.

→ Need Seiberg-Witten invariants.

Seiberg-Witten Invariants

Let X be symplectic w/ $\pi_1(X) = 0$ and $b_2^+ > 1$

- Seiberg-Witten invariants will be an integer-valued function

$$SW_X : H^2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}/_\pm$$

→ Nonzero at only finitely many classes of H^2.
("basic classes")

Basic Theorems:

Nonvanishing Thm: If X is symplectic then
$$SW_X(c_1(X)) = \pm 1$$
→ In particular $SW_X \not\equiv 0$

Vanishing Thm: If $X = X_1 \# X_2$ w/ $b_2^+(X_i) > 0$ then $SW_X \equiv 0$.

→ In particular X is not symplectic.

↳ These imply that $\#_n \mathbb{CP}^2$ is not symplectic. $n \geq 2$.

Symplectic Operations

- Symplecic Blowup
- Rational Blowdown
- Fibered Knot Surgery
- Fibered Sum (Gompf '95)

Let's look at Fibered Sum:

Let X_1, X_2 be symplectic manifolds & $\Sigma_i \subset X_i$ symplectic surfaces

w/ $e(N(\Sigma_1)) = -e(N(\Sigma_2))$

Euler-class of normal bdl

"self" intersection"

Def: The symplectic fiber sum is

$$X = X_1 \setminus N(\Sigma_1) \coprod_{\rho} X_2 \setminus N(\Sigma_2)$$

$$\rho : \partial N(\Sigma_1) \xrightarrow{\cong} \partial N(\Sigma_2) \quad \text{orientation reversing}$$

Thm: Diffeomorphism type of X does not depend on ρ.

EX: $E(1) = \mathbb{C}P^2 \# (\#_9 \overline{\mathbb{C}P^2})$ has a holomorphic Lefschetz fibration.

symplectic manifold (Kahler manifold in fact)

Regular fibers are tori.

$H^2 = \mathbb{Z}^{10}$ generators $\langle h \rangle \oplus \langle e_1 \rangle \oplus \cdots \oplus \langle e_9 \rangle$

$[F] = 3h - e_1 - \cdots - e_9 \qquad F^2 = 0$

$\Rightarrow e(N(F)) = 0$

- Fiber sum of two copies gives $E(2)$

$$E(2) = E(1) \#_{\text{fiber}} E(1)$$

④ $E(2)$ is interesting. It has intersection form

$$Q_{E(2)} = 2E_8 \oplus 3H_n$$

8×8

$$H_n = \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix} \qquad E_8 = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & \ddots & 1 & & 1 \\ & & 1 & -2 & 1 & 0 \\ & & & 1 & -2 & 0 \\ & & 1 & 0 & 0 & -2 \end{pmatrix}$$

→ Fiber sum w/ $E(1)$ gives $E(3)$

$$E(3) = E(2) \#_{\text{fiber}} E(1)$$

$E(3)$ has signature $\sigma = -24$ and Euler char. $e = 36$
w/ odd parity.

symplectic

Note: $E(3)$ homeom. to ~~…#~~ $(\#_5 \mathbb{C}P^2) \# (\#_{29} \overline{\mathbb{C}P^2})$

but $(\#_5 \mathbb{C}P^2) \# (\#_{29} \overline{\mathbb{C}P^2})$ is not symplectic

⇝ Example of homeom. but not diffeom. manifolds!

Next Time: Fibered Knot Surgery.