Özgür Kişisel — Roots of Unity & $u_q(sl(2))$ — 7/2/13 — ①

Recall: • (H, R) is a *quasitriangular bialgebra* if

① H is a bialgebra

② $R \in H \otimes H$ satisfies

$$\tau \circ \Delta h = R \cdot (\Delta h) \cdot R^{-1} \quad \text{for all } h$$

$$(\Delta \otimes id) R = R_{13} R_{23}$$

$$(id \otimes \Delta) R = R_{13} R_{12}$$

• $(S \otimes id) R = R^{-1}$

$(id \otimes S) R^{-1} = R$

$(S \otimes S) R = R$

• (H, R) is *triangular* if $Q = R_{21} \cdot R$ is trivial

• **Thm:** Let (H, R) be quasitriangular, Hopf then

① S is invertible

② $\exists$ invertible $u \in H$ w/

$$S^2(h) = u h u^{-1} \quad \text{all } h$$

$$\Delta u = Q^{-1}(u \otimes u) \quad \rightsquigarrow \left(u = (S R_2) R_1 \right)$$

Cor: (of theorem)

Let $v = Su$. Then v has the following properties:

① $S^{-2} h = v h v^{-1}$ all h

② $\Delta v = Q^{-1}(v \otimes v)$

③ $uv = vu$ is central

④ $\Delta(uv) = Q^{-2}(uv \otimes uv)$

⑤ uv^{-1} is group like $\left(\Delta(uv^{-1}) = uv^{-1} \otimes uv^{-1} \right)$

& implements S^4 by conjugation.

Proof:

$S^2 u = u u u^{-1} = u$ (from thm)

① $\mapsto S(v h v^{-1}) = S v^{-1} \, Sh \, Sv$

$= S(Su)^{-1} \, Sh \, S^2 u$

$= u^{-1} \, Sh \, u$

② $$\Delta v = \Delta S u = \Delta S((SR_1)R_2) = \Delta SR_2 R_1 = SR_2 \otimes R_1$$

$$Q^{-1}(v\otimes v) = R^{-1}\cdot R_{21}^{-1}\cdot (v\otimes v) = (R^{-1}\cdot R_{21}^{-1})\cdot(SR_1\cdot R_2 \otimes SR_1\cdot R_2) = ??$$

$$\Delta v = \Delta S u = \tau\circ(S\otimes S)\circ\Delta u = \tau\circ(S\otimes S)(Q^{-1}(u\otimes u)) = (v\otimes v)\cdot(\tau\circ(S\otimes S)Q^{-1}) = (v\otimes v)\cdot Q^{-1} \overset{?}{=} Q^{-1}(v\otimes v)$$

HOMEWORK !!

③ $$v^{-1}uv = S^2 u = u \rightsquigarrow uv = vu$$

$$\left.\begin{aligned} uvhv^{-1}u^{-1} &= u(S^{-2}h)u^{-1} \\ &= S^2S^{-2}h = h \end{aligned}\right\} \Rightarrow (uv)h = h(uv) \text{ central.}$$

④ $$\Delta(uv) = (\Delta u)\cdot(\Delta v) = Q^{-1}(u\otimes u)\cdot Q^{-1}(v\otimes v) \overset{?}{=} Q^{-2}(uv\otimes uv)$$

⑤ $$\Delta(uv^{-1}) = (\Delta u)\cdot(\Delta v^{-1}) = Q^{-1}(u\otimes u)\cdot(Q^{-1}(v\otimes v))^{-1} = Q^{-1}Q(uv^{-1}\otimes uv^{-1})$$

$$uv^{-1}h(uv^{-1})^{-1} = u(S^2h)u^{-1} = S^4 h$$

Def: A quasitriangular Hopf algebra is "<u>ribbon</u>" if uv has a central square root r ($r^2 = uv$) w/

① $\Delta r = Q^{-1}(r\otimes r)$

② $\eta r = 1$

③ $Sr = r$

(r is called the "<u>ribbon element</u>")

(This will be used later in applications to knot theory...)

Example:

$\mathbb{Z}/n$ = cyclic group of order n.

Let $H = \mathbb{C}\,\mathbb{Z}/n$ ~ the group algebra over $\mathbb{Z}/n$ as a Hopf algebra

$$\left[\begin{array}{l} \Delta g = g \otimes g \\ S g = g^{-1} \\ \varepsilon g = 1 \qquad (\text{for } g \in \mathbb{Z}/n) \end{array}\right.$$

→ This has two quasitriangular Hopf alg. structures
- trivial & nontrivial

"Nontrivial" structure is:

$$\boxed{R = \frac{1}{n} \sum_{a,b=0}^{n-1} q^{-ab} \cdot (g^a \otimes g^b) \quad \text{where} \quad q = e^{2\pi i/n}}$$

$$Q = \frac{1}{n} \sum_{a,b=0}^{n-1} q^{-ab} \cdot (g^{2a} \otimes g^b)$$

$$u = v = r = \frac{1}{n} \sum_{a,b=0}^{n-1} g^a \, \theta(a) \quad \text{where} \quad \theta(a) = \sum_{b=0}^{n-1} q^{-(a+b)b}$$

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* $\mathbb{Z}/n$ - theta funct.

Proof:

→ $\tau \circ \Delta h = R(\Delta h)R^{-1}$ is trivial b/c alg. is commutative & cocommutative

→ $(\Delta \otimes id)R = \frac{1}{n} \sum_{a,b=0}^{n-1} q^{-ab} (g^a \otimes g^a \otimes g^b)$

$$R_{13}R_{23} = \frac{1}{n^2} \sum_{a,b,c,d=0}^{n-1} q^{-(ab+cd)} \cdot (g^a \otimes g^c \otimes g^{d+b})$$

$$= \frac{1}{n} \cdot \sum_{a,c,d'=0}^{n-1} \left(\frac{1}{n} \sum_{b=0}^{n-1} q^{-b(a+c)} \cdot q^{-cd'} \cdot g^a \otimes g^c \otimes g^{d'} \right) \qquad d' = b+d$$

$$= \frac{1}{n} \sum_{a,d'=0}^{n-1} q^{-ad'} \cdot (g^a \otimes g^a \otimes g^{d'})$$

$$\boxed{\text{Note: } \frac{1}{n} \sum_{b=0}^{n-1} q^{ab} = \delta_{a,0}}$$

cancel

→ other is similar

→ $Q = R_{21} R = R^2$

$$= \frac{1}{n^2} \sum_{a,b,c,d} q^{-ab-cd} \cdot (g^{a+c} \otimes g^{b+d}) \qquad \begin{array}{l} a' = a+c \\ b' = b+d \end{array}$$

$$= \frac{1}{n^2} \sum_{a',b',c,d} q^{-(a'-c)(b'-c)-cd} \; g^{(a')} \otimes g^{(b')}$$

$$= \frac{1}{n} \sum_{a',b',c} \left(q^{-(a'-c)b'} \cdot (g^{a'} \otimes g^{b'}) \cdot \left(\frac{1}{n} \sum_{d'} q^{-d(2c-a)} \right) \right) \qquad \boxed{2c = a}$$

$= \cancel{[illegible]}$

$$= \frac{1}{n} \sum_{b',c}{}' \, q^{-(2c-c)b'} \, g^{2c} \otimes g^{b'}$$

$$= \frac{1}{n} \sum_{a,b}{}' \, q^{ab} \, g^{2a} \otimes b$$

$\rightarrow \; u = (S R_2) R_1$

$$= \frac{1}{n} \sum_{a,b=0}^{n-1}{}' \, q^{-ab} \cdot g^{-b} \cdot g^{a}$$

$$= \frac{1}{n} \sum_{a,b}{}' \, q^{-ab} \, g^{a-b}$$

$a' = a - b$

$$= \frac{1}{n} \sum_{a',b}{}' \, q^{-(a'+b)b} \, g^{a'}$$

$$= \frac{1}{n} \sum_{a'}{}' \, \theta(a') \, g^{a'} \qquad \text{w/} \qquad \theta(a') = \sum_{b}{}' \, q^{-(a'+b)b}$$
