# RESEARCH STATEMENT

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My research interests lie within differential geometry, geometric analysis and differential topology, primarily in the field of calibrations, calibrated geometries and manifolds with special holonomy. My current research interests include: plurisubharmonic functions and convexity in calibrated geometries, $\phi$-free submanifolds of calibrated manifolds, and manifolds with $G_2$ or $Spin(7)$ structures.

Working on different calibrated manifolds, I have obtained results on:

- geometry of $\phi$-critical submanifolds with some explicit examples
- topology of strictly $\phi$-convex domains in various calibrated manifolds
- topological restrictions for a submanifold to be $\phi$-free in some calibrated manifolds
- construction of manifolds with $Spin(7)$-holonomy
- existence of symplectic 8-manifolds with $Spin(7)$-structure.

I will start my statement with an introduction to some background material, then give a summary of my results with some details, and end with my present and future research projects.

## 1. Introduction

1.1. **Calibrated Geometries.** Calibrated geometries are introduced by Harvey and Lawson in the foundational paper [$HL_1$]. These are the geometries of minimal submanifolds of a Riemannian Manifold $(M, g)$ which are determined by a closed differential $p$-form $\phi$, called a *calibration*, which satisfies the property that $\phi_{|\xi} \leq vol_{|\xi}$ for any oriented tangent $p$-plane $\xi \subset T_xM$ at any point $x \in M$. A Riemannian manifold $(M, g)$ together with a calibration $\phi$ is called a *calibrated manifold* and any oriented $p$-dimensional submanifold $N$ of $M$ with $\phi|_N = vol_N$ is called a *calibrated submanifold* or *$\phi$-submanifold*. The fundamental observation here is that all calibrated submanifolds are minimal, and any compact oriented $\phi$-submanifold is volume minimizing in its homology class.

A Kähler form $\omega$ provides the basic and classical example of a calibration. In fact the form $\phi = \frac{\omega^p}{p!}$ for $1 \leq p \leq n$ on a Kähler manifold of dimension $n$ is a calibration, and $\phi$-submanifolds are precisely the $p$-dimensional complex submanifolds. In fact calibrated geometry is closely connected with the theory of Riemannian holonomy groups because Riemannian manifolds with special holonomy usually come equipped with one or more natural calibrations. This can be summarized in the following table using Berger's classification of holonomy groups.

| **Holonomy** | **Type of Manifold** | **Calibration** |
|---|---|---|
| $U(n)$ | Kähler Manifold | $\frac{\omega^p}{p!}$ ($\omega$=Kähler form) |
| $SU(n)$ | Calabi-Yau Manifold | Re($\Omega$) ($\Omega$=Holomorphic Volume form) |
| $Sp(n) \cdot Sp(1)$ | Quaternionic Kähler Manifold | $\Psi = \frac{1}{3}(\frac{\omega_I^2}{2} + \frac{\omega_J^2}{2} + \frac{\omega_K^2}{2})$ |
| $G_2$ | $G_2$-Manifold | $\varphi$ (associative 3-form) |
| $G_2$ | $G_2$-Manifold | $*\varphi$ (co-associative 4-form) |
| $Spin(7)$ | $Spin(7)$-Manifold | $\Phi$ (Cayley 4-form) |

Calibrated Geometries, especially on spaces with special holonomy, have strong relations with gauge theories in higher dimensions [AS], [T], mirror symmetry [SYZ], and modern string theory in Physics (cf. [$HL_1$], [$J_1$]). Hence, understanding the structure of these special geometries plays an important role and forms a very active and hot research area for geometers and physicists.

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1.2. **Plurisubharmonic Functions and Convexity on Calibrated Manifolds.** Analysis and geometry have always been very difficult on calibrated manifolds $(M, \phi)$ due to lack of analogues of holomorphic functions and curves existing in Kähler geometry. Recently, Harvey and Lawson brought in new tools, called *$\phi$-plurisubharmonic functions* to do analysis on any calibrated manifold by canonically generalizing the classical plurisubharmonic functions in Kähler manifolds to all calibrated manifolds. It turns out that these new functions exist in abundance and share properties similar to the classical ones in complex analysis.

In [HL$_2$] $\phi$-plurisubharmonic functions are defined by a second order differential operator $\mathcal{H}^\phi(f)$ called the ***$\phi$-Hessian*** which is defined from the set of smooth functions on $M$ to the set of smooth $p$-forms on $M$ by

$$\begin{aligned} \mathcal{H}^\phi : C^\infty(M) \ &\rightarrow \ \mathcal{E}^p(M) \\ \mathcal{H}^\phi(f) \ &= dd^\phi f - \nabla_{\nabla f}(\phi) \end{aligned}$$

where $d$ is the de Rham differential, and $d^\phi : C^\infty(M) \ \rightarrow \ \mathcal{E}^{p-1}(M)$ is given by

$$d^\phi f \ \equiv \ \nabla f \lrcorner \phi.$$

If $\phi$ is parallel, then we get $\mathcal{H}^\phi = dd^\phi$

On a calibrated manifold $(M, \phi)$, an oriented tangent $p$-plane $\xi \subset T_xM$ at $x \in M$ is called a *$\phi$-plane* if $\phi|_\xi = vol_\xi$ and the set of all $\phi$-planes is denoted by $G(\phi)$. A function $f \in C^\infty(M)$ is defined to be **$\phi$-plurisubharmonic** if $\mathcal{H}^\phi(f)(\xi) \geq 0$ for all $\xi \in G(\phi)$. It is **strictly $\phi$-plurisubharmonic** at a point $x \in M$ if $\mathcal{H}^\phi(f)(\xi) > 0$ for all $\phi$-planes $\xi$ at $x$. We denote the set of $\phi$-plurisubharmonic functions on a calibrated manifold $M$ by $\mathcal{PSH}(M, \phi)$. In the Kähler case, $\phi$-plurisubharmonic functions are exactly the classical plurisubharmonic functions since it is easy to show that $dd^\omega = dd^c$ and $\omega$-planes are complex lines.

A fundamental result is that *the restriction of a $\phi$-plurisubharmonic function to a $\phi$-submanifold $N$ is subharmonic in the induced metric on $N$.*

One of the first outcomes of introducing these canonical new functions on calibrated manifolds is *$\phi$-convexity*, which is the generalization of pseudoconvexity on complex manifolds to calibrated manifolds. A calibrated manifold $(M, \phi)$ is called **(strictly) $\phi$-convex** if it admits a (strictly) $\phi$-plurisubharmonic proper exhaustion function $f : M \rightarrow \mathbf{R}$. We note that in complex geometry with Kähler calibration $\omega$, strictly $\omega$-convex manifolds are Stein.

1.3. **$\phi$-free Submanifolds.** In any calibrated manifold, the $\phi$-free submanifolds are the analogues of totally real submanifolds in complex manifolds. Just as totally real submanifolds are used to construct Stein manifolds with different homotopy type, $\phi$-free submanifolds are used to construct strictly $\phi$-convex manifolds in enormous families with every topological type allowed by Morse theory.

Let $(M, \phi)$ be a calibrated manifold. A closed submanifold $N \subset M$ is called **$\phi$-free** if there are no *$\phi$-planes* tangential to $N$ i.e. no $\xi \in G(\phi)$ with $span\xi \subset TN$. If the calibration is a $p$-form, then obviously any submanifold of dimension < p is $\phi$-free. Generically, this is true for local submanifolds of dimension $p$.

The **$\phi$-free dimension** of a calibrated manifold $(M, \phi)$, denoted by **fd($\phi$)**, is defined to be the largest dimension of a $\phi$-free vector subspace of $T_xM$ for $x \in M$, which also determines the maximum possible dimension of a $\phi$-free submanifold. A fundamental result, following this definition, is that *any strictly $\phi$-convex calibrated manifold $(M, \phi)$ has the homotopy type of a CW-complex of dimension $\leq$ fd($\phi$).* This result is a generalization of the Andreotti-Frankel Theorem [AF] of Stein manifolds.

For a Kähler manifold $(M, \omega)$ of complex dimension $n$, fd$(\omega) = n$ and $\omega$-free submanifolds are totally real. For a quaternionic Kähler manifold $(M, \Psi)$ of real dimension $4n$, where $\Psi = \frac{1}{6}\{w_I^2 + w_J^2 + w_K^2\}$ we get fd$(\Psi) = 3n$.

If $(M, \varphi)$ is a 7-dimensional $G_2$-manifold with associative calibration $\varphi$, then fd($\varphi$)=4. Moreover, the free dimension of coassociative calibration $*\varphi$ is also equal to 4.

If $(M, \Phi)$ is an 8-dimensional $Spin(7)$-manifold with Cayley calibration $\Phi$, then fd($\Phi$)=4.

## 2. Past Research

2.1. **Topology of $\phi$-Convex Manifolds and $\phi$-Free Submanifolds.** (cf. [U], [$U_1$]) In [$HL_1$] Harvey and Lawson show that *if $N$ is a $\phi$-free submanifold of $(M,\phi)$, then there exists a fundamental system $\mathcal{F}(N)$ of strictly $\phi$-convex neighborhoods of $N$, each of which admits a deformation retraction onto $N$.* Hence the existence of $\phi$-free submanifolds with different homotopy types ensures the existence of lots of strictly $\phi$-convex domains in $(M,\phi)$. Using techniques from differential topology, I studied the existence of $\phi$-free submanifolds in various calibrated manifolds and obtained results about their topology. Moreover, in certain cases I constructed examples with different homotopy types.

On a quaternionic Kähler or HyperKähler manifold $(M^{4n},\Psi)$ with the quaternionic calibration $\Psi=\frac{1}{6}\{w_I^2+w_J^2+w_K^2\}$ I showed that *arbitrary small $C^r$-perturbations (r>0) of any k-dimensional closed oriented submanifold $N\subset M$ in $M$ will be $\Psi$-free if $5k<12n+4$. In particular, if $M=\mathbb{H}^n$, then almost all Euclidean motions will make $N$ $\Psi$-free.* Moreover, I constructed a closed oriented $\Psi$-free submanifold of $\mathbb{H}^n$ with dimension $3n$, which proves the existence of a strictly $\Psi$-convex domain $\Omega$ with $H_{3n}(\Omega,\mathbb{Z})\neq 0$, where $\mathrm{fd}(\Psi)=3n$.

On a $G_2$-manifold $(M,\varphi)$ with associative calibration $\varphi$, I showed that *for every connected submanifold $N\subset M$ of dimension <4 (compact or non-compact) there exists a strictly $\phi$-convex $G_2$-manifold which is homotopy equivalent to $N$.*

On a $G_2$-manifold $(M,*\varphi)$ with coassociative calibration $*\varphi$, I proved that *if $N\subset M$ is $*\varphi$-free, then the Euler Characteristic of $N$, $\chi(N)$ is equal to zero.* This shows that the Euler Characteristic is an obstruction for the existence of a $*\varphi$-embedding of a manifold into a $G_2$-manifold. Recently, using $h$-principle I showed that this is the only obstruction as I will explain in the following section.

2.2. **$\phi$-Critical Submanifolds.** (cf. [U], [$U_2$]) Calibrated geometry is interesting and rich if we have lots of calibrated submanifolds, but this is not always the case, not even if the set of $\phi$-planes $G(\phi)$ is large. In [$HL_2$] Harvey and Lawson canonically extend their definition of $\phi$-submanifolds to $\phi$-critical submanifolds in such a way that every $\phi$-submanifold is $\phi$-critical. Despite the lack of non-trivial examples of $\phi$-submanifolds, we still get a very rich geometry if we consider all $\phi$-critical submanifolds. In my thesis [U] and later in my paper [$U_2$], I study the geometry of $\phi$-critical submanifolds, where I found non-trivial examples and proved an important result about their minimality.

If $\phi$ is a calibration on $M$, then we see that $\phi: G_k^+(T_xM)\longrightarrow[-1,1]$, for each $x\in M$, where $G_k^+(T_xM)$ is the Grassmannian of oriented $k$-planes in $T_xM$. Let us denote the critical points of $\phi_{|G(k,T_xM)}$ by $G^{cr}(\phi)_x$ and the associated sub-bundle in Grassmannian by $G^{cr}(\phi)$. Then, any oriented $k$-dimensional submanifold $N\subset M$ is called a **$\phi$-critical submanifold** with critical value $c$ if $T_xM\in G_c^{cr}(\phi)$, for all $x\in M$ where $G_c^{cr}(\phi)=\{\xi\in G^{cr}(\phi):\phi(\xi)=c\}$. As a result, $\phi$-submanifolds are $\phi$-critical submanifolds with critical value 1 since $G(\phi)=G_1^{cr}(\phi)$

If we consider $\mathbb{H}^n$ with quaternionic calibration $\Psi=\frac{1}{6}\{w_I^2+w_J^2+w_K^2\}$, then $\phi$-submanifolds are just quaternion lines. But, I showed that $\pm\frac{1}{3}$ *are critical values of $\Psi$, and $\Psi$-critical submanifolds include complex isotropic submanifolds for any complex structure defined by right multiplication by a unit imaginary quaternion. In particular for n=2 they include all complex Lagrangian submanifolds.*

We know that $\phi$-submanifolds are minimal and any compact oriented $\phi$-submanifold is homologically volume minimizing. In [$U_2$] I prove that *if a non-zero positive (negative) critical value $c$ of a calibration $\phi$ is a local maximum(minimum), then $\phi$-critical submanifolds with critical value $c$ are minimal and locally volume minimizing.* One corollary of this result is that we may find locally volume minimizing submanifolds by looking for local maximum points of a closed form $\phi$ on a compact manifold without ever computing the comass of $\phi$, which is usually very difficult.

2.3. **Warped-like Product 8-manifolds with $Spin(7)$-holonomy.** (cf. [UU]) In [YO] Yasui and Ootsuka give an explicit example of a warped-like metric with $Spin(7)$-holonomy on $M=S^3\times S^3\times\mathbb{R}^2$. Later in [BU], Bilge and Uğuz consider any warped-like metric $g$ on $M=M_1^3\times M_2^3\times\mathbb{R}^2$, where $M_1$ and $M_2$ are complete, simply connected 3-manifolds with metrics $g_1$ and $g_2$ respectively and show that if $(M, g)$ has holonomy $Spin(7)$, then each $(M_i, g_i)$ is isometric to $S^3$ with constant positive sectional curvature and up to a gauge transformation, the only solution is the explicit metric on $M = S^3 \times S^3 \times \mathbb{R}^2$ given by Yasui and Ootsuka.

In [UU] with S. Uğuz, I consider similar warped-like metrics on $M = M_1^4 \times M_2^3 \times \mathbb{R}$, where $M_1^4$ is a simply connected parallelizable 4-manifold with metric $g_1$, and $M_2^3$ is a simply connected 3-manifold with $g_2$ and study when the metric has $Spin(7)$-holonomy. We show that there will be two cases. In the first case, $M$ is isometric to $M = M_1^3 \times \mathbb{R} \times M_2^3 \times \mathbb{R}$, which is actually isometric to the case studied by Bilge and Uğuz. In the second case, we prove that $(M_1^4, g_1)$ *must have constant negative sectional curvature and* $(M_1^3, g_2)$ *is isometric to* $S^3$ *with constant positive sectional curvature. Moreover, by considering the structure equations, we conclude that* $(M_1^4, g_1)$ *is a solvable and non-unimodular Lie group with a left invariant metric.*

## 3. Present and Future Research Projects

3.1. **$h$-Principle and $\phi$-Free Embeddings.** Homotopy principle ($h$-principle), developed by M. Gromov, Y. Eliashberg, et al. is a very powerful method to reduce existence problems in differential geometry and partial differential equations to homotopy-theoretic problems (cf. [G], [EM]). One common application of the $h$-principle is to prove the existence of special embeddings. M. Gromov proves that the $h$-principle holds for totally real embeddings [G]. Since the generalization of totally real submanifolds to calibrated manifolds are precisely the $\phi$-free submanifolds, I study the $h$-principle for $\phi$-free embeddings.

In [$U_3$], by expressing the $\phi$-free condition in the terminology of the $h$-principle, namely as a differential relation, I prove that *all forms of the h-principle hold for* $*\varphi$*-free and* $\Phi$*-free embeddings of oriented 4-dimensional manifolds* $N^4$ *into* $\mathbb{R}^7$ *and into* $\mathbb{R}^8$ *respectively, where* $*\varphi$ *is the coassociative and* $\Phi$ *is the Cayley calibration. Moreover, all forms of the h-principle hold for* $\Psi$*-free embeddings of oriented k-dimensional manifolds* $N^k$ *into* $\mathbb{H}^n$*, where* $k \leq 3n$ *and* $\Psi$ *is the quarternionic calibration.* In addition to these, I prove that *a closed oriented 4-manifold* $N^4$ *can be embedded as a* $*\varphi$*-free or* $\Phi$*-free into* $\mathbb{R}^7$ *or* $\mathbb{R}^8$ *respectively if the Euler characteristic of* $N$ *is equal to zero.*

In [$U_3$] we only prove existence results about $\phi$-free embeddings. My future plan is to find the example of an explicit $*\varphi$-free embedding of a closed oriented 4-manifold into $\mathbb{R}^7$. My first candidate to try is $S^1 \times S^3$, whose canonical embedding into $\mathbb{R}^7$, i.e. $f : S^1 \times S^3 \hookrightarrow \mathbb{R}^2 \times \mathbb{R}^4 \times \mathbb{R}$ is not $*\varphi$-free.

3.2. **Examples of $\phi$-Critical Submanifolds.** In [$U_2$] I show that if the positive(negative) critical value of the calibration $\phi$ is a local maximum(minimum), then corresponding $\phi$-critical submanifolds are locally volume minimizing, hence minimal. Recently in [R], C. Robles shows that if $\phi$ is a parallel calibration, then the $\phi$-critical submanifolds corresponding to a non-zero critical value are minimal. From these results, we see that the $\phi$-critical submanifolds are a very good source of minimal submanifolds, especially in higher codimension.

In [U] I prove that $\pm\frac{1}{3}$ are critical values of the quaternionic calibration $\Psi = \frac{1}{6}\{w_I^2 + w_J^2 + w_K^2\}$ on $\mathbb{H}^n$ and for $n = 2$, the set of all $\Psi$-critical planes with critical value $\pm\frac{1}{3}$ are complex Lagrangian 4-planes with any complex structure defined by an imaginary quaternion. I also show that $\pm\frac{1}{3}$ are neither local maximum nor local minimum, hence they don't satisfy the criterion I proved. However, $\Psi$ is parallel so the corresponding $\Psi$-critical submanifolds are minimal by the criterion of Robles. (In [LW] these $\pm\frac{1}{3}$ $\Psi$-critcal submanifolds are actually called as *hyperlagrangian*.) In [U] I prove all my results about the quaternionic calibration $\Psi = \frac{1}{6}\{w_I^2 + w_J^2 + w_K^2\}$ on $\mathbb{H}^n$ using elementary methods which take a lot of calculations. Currently I am writing a shorter proof of these results which also works on any 8-dimensional quaternionic Kähler manifold and also trying to find more examples of hyperlagrangian submanifolds [$U_5$].

In [$U_2$] I study only one example of $\phi$-critical geometry, but I want to examine more examples. J.Zhou in [Z] shows that most of the well-known calibrations (Kähler, special Lagrangian, associative, coassociative and Cayley) don't have critical values other than +1 and -1. Hence in these cases,

$\phi$-critical geometry is the same as calibrated geometry. Therefore, I plan to continue investigating the family of calibrations $\Psi(\lambda) = \lambda_1 \frac{1}{2} w_I^2 + \lambda_2 \frac{1}{2} w_J^2 + \lambda_3 \frac{1}{2} w_K^2$ on $\mathbb{H}^n$, where $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ lies in the convex body defined by $\mid \lambda_j \mid \leq 1,\ j = 1, 2, 3$ and $\mid \lambda_1 + \lambda_2 + \lambda_3 \mid \leq 1$. I think this family is a good source for critical values different than -1 and +1 since the calibration I studied in [$U_2$], $\Psi = \frac{1}{6}\{w_I^2 + w_J^2 + w_K^2\}$, is just a special case of this family where $\lambda = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Details about this family of calibrations are available in [BH].

3.3. **Symplectic 8-manifolds with $Spin(7)$-structure.** (with S. Uğuz) Given a smooth manifold $M$, one can ask for the existence of two different structures and their compatibility. For any smooth 7-dimensional manifold $M$, Arikan, Cho, and Salur studied these problems for contact and $G_2$ structures, and found examples admitting both contact and $G_2$ structures which were compatible in a certain way [ACS]. There is a similar existence and compatibility problem for 8-dimensional smooth manifolds for symplectic and $Spin(7)$ structures. I have been studying this problem with Uğuz.

In [P] F. Pasquotto shows that in dimension 8 the geography of symplectic manifolds doesn't differ from that of almost complex manifolds by proving that any ordered quintuple of integers which are admissible as the system of Chern numbers of an almost complex 8-dimensional manifold can also be realized by a closed, connected symplectic 8-manifold.(Hirzebruch showed the necessary and sufficient conditions for a given system of integers to appear as the system of Chern numbers of an almost complex manifold by using the Riemann-Roch theorem [H], and these conditions are given by three relations in dimension 8.) Moreover, the existence of a $Spin(7)$-structure on an oriented 8-dimensional manifold $M$ is determined by the second Stiefel-Whitney class and an equation involving Pontryagin classes and the Euler class which can also be written in terms of Chern classes for an almost complex manifold. In [$U_3$] we show that three relations for the existence of symplectic structure together with two equations for the existence of $Spin(7)$-structure have solutions which proves the existence of many closed, connected symplectic manifolds with $Spin(7)$-structure without needing any compatibility. By studying the examples of Pasquotto, we can construct lots of examples. As a future project, we plan to study compatibility between these two structures in certain ways like it was studied in [ACS].

3.4. **Cotton Flow.** (with A. U. Ö. Kişisel) In [KST] Kişisel, Sarıoğlu and Tekin define the Cotton flow which flows the initial metric in a direction proportional to Cotton-York tensor on 3-dimensional manifolds and whose fixed points are conformally flat metrics. They study the evolution of the nine homogeneous geometries ($\mathbb{R}^3$, $H^3$, $H^2 \times \mathbb{R}$, $S^2 \times \mathbb{R}$, $SU(2)$, $Isom(\mathbb{R}^2)$, $Solv$, $Nil$, $SL(2, \mathbb{R})$) in detail, both analytically and numerically. However, they didn't prove the short time existence of the flow in the general case. Together with Kişisel, I plan to determine the conditions on the initial metric for which the short time existence problem can be solved. We have been studying the solutions of the short time existence problem for other flows, especially higher order flows, since the Cotton flow is third order and techniques in this order are not as developed as in the second order case. Moreover, we are also interested in finding Cotton solitons and this problem seems a little bit easier than the short time existence problem.

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