M E T U
Northern Cyprus Campus

	Basic Linear Algebra I. Midterm	
Code : *Math 260* Acad.Year: *2011-2012* Semester : *Spring* Date : *20.3.2012* Time : *17:40* Duration : *150 minutes*	Last Name: Name : ICEY Department: Signature: 9 QUESTIONS ON 8 PAGES TOTAL 100 POINTS	Student No: Section:

1 (10)	2 (12)	3 (10)	4 (10)	5 (12)	6 (12)	7 (12)	8 (12)	9 (10)		

1. (10 points) Let V be the set of all ordered pairs (a, b) of real numbers. Define a new addition and scalar multiplication in the following way:

$$(a, b) \oplus (c, d) = (a + c, b + d)$$

$$r \cdot (c, d) = (0, rd)$$

Is V with these operations a vector space? Explain your answer.

The set V is not a vector space equipped with these operations. Indeed, if it were a vector space, we would have

$$(1,0) = 1 \cdot (1,0) = (0, 1\cdot 0) = (0,0) = \vec{0},$$

a contradiction.

2. (12 points) Let $V = C[\overset{-1}{0}, 1]$ be the vector space of all continuous functions on the interval $[0, 1]$. Let E be the subset of V defined by

$$E = \{f(x) | f(x) \in C[\overset{-1}{0}, 1] \text{ and } f(-x) = -f(x), x \in [\overset{-1}{\cancel{0}}, 1]\}.$$

Show that E is a subspace of V (or $E \leq V$).

Take $f, g \in E$, $\lambda \in \mathbb{R}$. Then

$(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f+g)(x)$,

$(\lambda f)(-x) = \lambda \cdot f(-x) = -\lambda f(x) = -(\lambda f)(x)$

for all $x \in [0, 1]$. Therefore

$f + g \in E$ and $\lambda \cdot f \in E$.

3. (10 points) Is the following collection of vectors in $\mathbb{R}^3$ a linear subspace?

$$\mathcal{U} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 + x_3 \geq 0\}$$

The set $\mathcal{U}$ is not a subspace in $\mathbb{R}^3$, for

$\vec{a} = (1, 1, 1) \in \mathcal{U}$ whereas $-\vec{a} = (-1, -1, -1) \notin \mathcal{U}$.

4. (10 points) Suppose V is a vector space and E, F are subsets of V. Show that $Span(E \cup F) = Span(E) + Span(F)$ (or $\mathcal{L}(E \cup F) = \mathcal{L}(E) + \mathcal{L}(F)$).

Based on the trivial inclusions $E \subseteq E \cup F$ and $F \subseteq E \cup F$, we have $Span(E) \subseteq Span(E \cup F)$, $Span(F) \subseteq Span(E \cup F)$.

Being $Span(E \cup F)$ a subspace in V, we derive that

$$Span(E) + Span(F) \subseteq Span(E \cup F).$$

Conversely, pick $x \in Span(E \cup F)$. By its very definition $x = \sum_{i=1}^{n} \lambda_i e_i + \sum_{j=1}^{m} \mu_j f_j$ for some $\{e_i\} \subseteq E$, $\{f_i\} \subseteq F$ and $\{\lambda_i, \mu_j\} \subseteq \mathbb{R}$.

But $x_1 = \sum_{i=1}^{n} \lambda_i e_i \in Span(E)$ and

$$x_2 = \sum_{j=1}^{m} \mu_j f_j \in Span(F).$$

Whence $x = x_1 + x_2 \in Span(E) + Span(F)$, that is,

$$Span(E \cup F) \subseteq Span(E) + Span(F).$$

5. (12 points) Let $\mathcal{U}$ be the subspace of $\mathcal{P}_3(\mathbb{R})$ spanned by

$$E = \{2x^3 + 1, 5, x^3, x^2 + 1, x^3 - 3\}.$$

Find a linearly independent subset F of E spanning $\mathcal{U}$, that is, $\mathcal{U} = Span(E) = Span(F)$ (or $\mathcal{U} = \mathcal{L}(E) = \mathcal{L}(F)$).

First note that

$$2x^3 + 1 = 2 \cdot x^3 + \frac{1}{5} \cdot 5 \in Span\{x^3, 5\},$$

$$x^3 - 3 = 1 \cdot x^3 + \left(-\frac{3}{5}\right) \cdot 5 \in Span\{x^3, 5\}$$

Put $F = \{5, x^3, x^2+1\} \subseteq E$. If

$\lambda_1 \cdot 5 + \lambda_2 x^3 + \lambda_3 (x^2+1) = 0$ then we have

$(5\lambda_1 + \lambda_3) + \lambda_3 x^2 + \lambda_2 x^3 = 0$ or

$$\begin{cases} 5\lambda_1 + \lambda_3 = 0 \rightarrow \lambda_1 = 0 \\ \lambda_3 = 0 \\ \lambda_2 = 0 \end{cases}$$

Hence F is a linearly independent subset of E such that

$$Span(F) = Span(E) = \mathcal{U}.$$

6. (12 points) Show that the functions $\{e^x, e^{-x}, \sin x\}$ are linearly independent in the vector space $C[-\pi, \pi]$.

Assume $\lambda_1 e^x + \lambda_2 e^{-x} + \lambda_3 \sin(x) = 0$

for some $\lambda_i \in \mathbb{R}, 1 \leq i \leq 3$.

$$\left.\begin{aligned} x=0 &\longrightarrow \lambda_1 + \lambda_2 = 0 \\ x=\pi &\longrightarrow \lambda_1 e^{\pi} + \lambda_2 e^{-\pi} = 0 \end{aligned}\right\} \Rightarrow \lambda_1 = \lambda_2 = 0$$

It follows that $\lambda_3 \sin(x) = 0$.

$$x = \frac{\pi}{2} \rightarrow \lambda_3 = \lambda_3 \sin\left(\frac{\pi}{2}\right) = 0.$$

Thus $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

7. (12 points) (a) Find a basis for the following subspace of $\mathbb{R}^3$:

$$\mathcal{T} = \{(x, y, z) | 2x - y + 5z = 0\}.$$

Put $\vec{a} = (1, 2, 0)$, $\vec{b} = (0, 5, 1) \in \mathcal{T}$. If $\lambda\vec{a} + \mu\vec{b} = \vec{0}$ then $(\lambda, 2\lambda + 5\mu, \mu) = \vec{0} \Rightarrow \lambda = \mu = 0$.
Therefore, $3 \geq \dim(\mathcal{T}) \geq 2$. But $\vec{n} = (2, -1, 5) \notin \mathcal{T}$, that is, $\mathcal{T} \neq \mathbb{R}^3$. It follows that $\dim(\mathcal{T}) = 2$. In particular, $\mathcal{T} = \text{Span}\{\vec{a}, \vec{b}\}$ or $\{\vec{a}, \vec{b}\}$ is a basis for $\mathcal{T}$.

(b) Complete the basis found in part (a) to a basis of $\mathbb{R}^3$.

Since $\vec{n} \notin \text{Span}\{\vec{a}, \vec{b}\}$, it follows that $\{\vec{a}, \vec{b}, \vec{n}\}$ are linearly independent vectors in $\mathbb{R}^3$. But $\dim(\mathbb{R}^3) = 3$, thereby $\{\vec{a}, \vec{b}, \vec{n}\}$ is a basis for $\mathbb{R}^3$.

9. (10 points) Let $S = \{1, 2, \ldots, 100\}$ and $T = \{99, 100\} \subseteq S$. Find a basis for $Fun(S, T)$. (Recall that $Fun(S, T) = \{f \in Fun(S) | f(t) = 0, \forall t \in T\}$.) (Hint: Use the characteristic functions $\chi_s, s \in S$.)

We know that $X = \{\chi_n : 1 \leq n \leq 100\}$ is a basis for $Fun(S)$. Moreover,

$$\{\chi_n : 1 \leq n \leq 98\} \subseteq Fun(S, T)$$

is a linearly independent subset.

Pick $f \in Fun(S,T)$. Then $f(99) = f(100) = 0$

and

$$f = \sum_{s \in S} f(s)\chi_s = \sum_{n=1}^{100} f(n)\chi_n = \sum_{n=1}^{98} f(n)\chi_n \in$$
$$\in Span\{\chi_n : 1 \leq n \leq 98\}$$

Thus $Fun(S,T) = Span\{\chi_n : 1 \leq n \leq 98\}$ and $\{\chi_n : 1 \leq n \leq 98\}$ is a basis for $Fun(S,T)$, and $\dim(Fun(S,T)) = 98$.