Solutions

METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY
MIDTERM EXAMINATION

Code : *MAT 119*	Last Name:
Acad.Year: *2014-2015*	Name : Student No.:
Semester : *SUMMER*	Department: Section:
Date : *14.7.2015*	Signature:
Time : *17:40*	5 QUESTIONS ON 6 PAGES
Duration : *120 minutes*	TOTAL 100 POINTS

1. (24)	2. (20)	3. (20)	4. (20)	5. (16)		

Show your work! Please draw a box around your answers!

1. (8, 8, 8) Calculate the following derivatives.

(A) $\frac{d}{dx}\left(x^7\tan\left(\sqrt{x^2-6}\right)\right)$

$$= \boxed{7x^6\tan\left(\sqrt{x^2-6}\right) + x^7\sec^2\left(\sqrt{x^2-6}\right)\frac{1}{2\sqrt{x^2-6}}\,2x}$$

(B) $\frac{d}{dx}\left(\frac{x\sin(x)}{1+x}\right)$

$$= \boxed{\frac{\left(1\sin(x) + x\cos(x)\right)(1+x) - x\sin(x)\cdot 1}{(1+x)^2}}$$

$$= \frac{x^2\cos(x) + x\cos(x) + \sin(x)}{(1+x)^2} = \frac{x\cos(x)}{1+x} + \frac{\sin(x)}{(1+x)^2}$$

(C) $\frac{d}{dx}\left(\cos^{107}\left(\sin(x+3)\right)\right)$

$$= \boxed{107\cos^{106}\left(\sin(x+3)\right)\left(-\sin\left(\sin(x+3)\right)\right)\cos(x+3)}$$

2. (10 + 10) Let $f(x) = \dfrac{x+1}{x-1}$.

(A) Find the equation of the tangent line to the graph of this function at the point (2,3).

$f'(x) = \dfrac{-2}{(x-1)^2}$ hence the slope of tangent at (2,3) is $f'(2) = -2$

Consequently, the equation of the tangent in question is

$$\boxed{y - 3 = -2(x-3)}$$

upon regrouping

$$2x + y - 7 = 0 .$$

(B) Find the points on this curve at which the normal lines are parallel to the line

$$9x - 2y = 381 .$$

The slope of the normal at a point $(x, f(x))$ on the curve is $-1/f'(x)$. Therefore, in order for such a normal to be parallel to the given line it is necessary & sufficient that

$$-\frac{1}{f'(x)} = \frac{(x-1)^2}{2} = 9/2$$

hence $x - 1 = \pm 3$. It follows that the required points are

$$\boxed{P(4, 5/3) \ \& \ Q(-2, 1/3)}$$

an assumption justified below by $50+12\cdot5\cdot1=110\neq 0$!

3. (10 + 10)

(A) Find the equation of the tangent line to the curve given by the equation

$$3(x^2+y^2)^2 = 25(x^2-y^2)$$

at the point (2, 1).

Assuming that this equation near (2,1) can be solved for y in terms of x the derivative $y'(x)$ can be found by taking the derivative of both sides with respect to x & observing

$$3\cdot 2\,(x^2+y^2)\,(2x+2yy') = 25\,(2x-2yy')$$

hence $\quad 12\,(x^2+y^2)\,x + 12\,(x^2+y^2)\,yy' = 50x - 50yy'.$

Thus, the slop of the tangent required is found to be

$$y'(2) = \left.\frac{50x-12(x^2+y^2)x}{50y+12(x^2+y^2)y}\right|_{x=2,\,y=1} = \frac{100-12\cdot5\cdot2}{50+12\cdot5\cdot1} = -\frac{2}{11}.$$

The required tangent is

$$\boxed{y-1=-\frac{2}{11}(x-2).}$$

or

$$2x+11y-15=0$$

(B) Let the tangent line at <u>any</u> point on the curve given by the equation

$$x^{2/3}+y^{2/3}=1$$

intersect the y-axis in the point P and the x-axis in the point Q. Prove that $|PQ|=1$.

Given an arbitrary point (a,b) on the curve – so that $a^{2/3}+b^{2/3}=1$ – the slope of the tangent is found by

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0 \longrightarrow y' = -\frac{x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3} \quad \& \quad m = y'(a) = -\left(\frac{b}{a}\right)^{1/3}$$

provided $a\neq 0$.

The tangent line is $\quad y-b = -\left(\frac{b}{a}\right)^{1/3}(x-a)$. Hence

$$x=0 \longrightarrow y = b + b^{1/3}a^{2/3} = b^{1/3} \Longrightarrow P(0, b^{1/3})$$

$$y=0 \longrightarrow x = a + a^{1/3}b^{2/3} = a^{1/3} \Longrightarrow Q(a^{1/3}, 0).$$

Therefore $\quad |PQ|^2 = a^{2/3}+b^{2/3} = 1.$

The case with $a=0$ (which implies $b\neq 0$) can be taken care of by interchanging the roles of x and y !

4. (10 + 10)

(A) Find the largest and the smallest value of

$$f(x) = \frac{x}{x^2 - x + 2}$$

on the interval $[-1, 3]$.

(Note first, that $x^2 - x + 2 = (x - \frac{1}{2})^2 + \frac{3}{4} \neq 0$. Thus the domain of f contains $[-1, 3]$. & f is continuous on $[-1, 3]$)

$$f'(x) = \frac{-x^2 + 2}{(x^2 - x + 2)^2} = 0 \longrightarrow x = \sqrt{2}$$ as the only critical point in $[-1, 3]$.

Check the critical points & the end points:

$f(-1) = \boxed{-\frac{1}{4}}$ The least value of f.

$f(\sqrt{2}) = \frac{\sqrt{2}}{4 - \sqrt{2}} = \boxed{\frac{2\sqrt{2} + 1}{7}}$ The largest value of f.

$f(3) = \frac{3}{8}$

(B) Employing the mean value theorem or otherwise, show that

$$\sin(x) < x$$

for all $x > 0$.

By the MVT

$\sin(x) - \underbrace{\sin(0)}_{0} = \cos(c)(x - 0)$ for some $c \in (0, x)$.

As $\cos(c) < 1$ for $x \leq 2\pi$

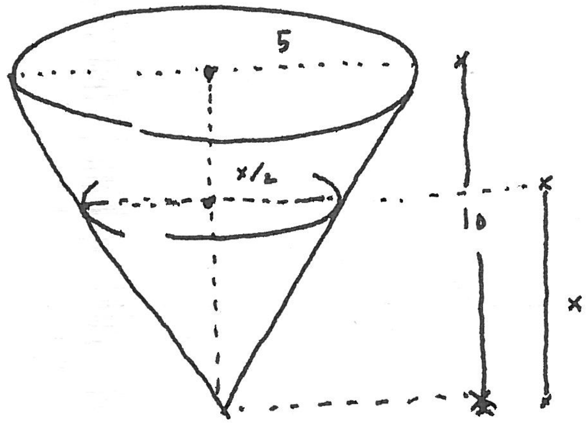
$\sin(x) < x$.

If $x > 2\pi$, the inequality is voidly satisfied since $\sin(x) \leq 1$!

5. (16)

Water leaks at a constant rate from a tank in the shape of an inverted right circular cone of height 10m and base radius 5m. If the water level is observed to sink at the rate of 10 cm/hour when the tank is full, what is the rate at which the water level will sink when the water is 8m deep ?

The volume of the water when the depth is x, is



$$V(x) = \frac{1}{3}\pi\left(\frac{x}{2}\right)^2 \cdot x = \frac{\pi}{12}x^3.$$

The constant rate of leak is

$$\frac{dV}{dt} = \frac{\pi}{4}x^2 \left.\frac{dx}{dt}\right|_{x=10} = \frac{\pi}{4}100\cdot(-0{,}1) = -\frac{5\pi}{2}\ m^3/h.$$

time

When $x = 8$ m

$$-\frac{5\pi}{2} = \frac{\pi}{4}64\cdot\frac{dx}{dt} \longrightarrow \frac{dx}{dt} = \boxed{-\frac{5}{32}\ m/\text{hour}}$$

~~thm~~.

& the water level sinks at the rate of $5/32$ ~~m~~ m/h

or $\simeq$ ~~...~~ 15 cm per hour.