Solutions

METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY
FINAL EXAMINATION

Code : *MAT 119*	Last Name:
Acad.Year: *2014-2015*	Name : Student No.:
Semester : *SUMMER*	Department: Section:
Date : *14.8.2015*	Signature:
Time : *09:00*	5 QUESTIONS ON 6 PAGES
Duration : *120 minutes*	TOTAL 100 POINTS

1. (24)	2. (20)	3. (24)	4. (16)	5. (16)			

Show your work! Please draw a $\boxed{\text{box}}$ around your answers!

1. (7 + 8 + 9) Compute the following integrals.

(A) $\displaystyle\int \frac{x-2x^3}{\sqrt[3]{x}}\,dx$ $\displaystyle = \int \left(x^{2/3} - 2x^{8/3}\right)dx = \frac{x^{5/3}}{5/3} - 2\,\frac{x^{11/3}}{11/3}$

$$= \boxed{\frac{3}{5}x^{5/3} - \frac{6}{11}x^{11/3}} + C$$

$du = 3x^2\,dx$

$u = x^3 - 1$

(B) $\displaystyle\int x^5\sqrt{x^3-1}\,dx$ $\displaystyle = \int (u+1)\sqrt{u}\cdot\frac{1}{3}\,du = \frac{1}{3}\left[\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right]$

$$= \boxed{\frac{2}{15}(x^3-1)^{5/2} + \frac{2}{9}(x^3-1)^{3/2}} + C$$

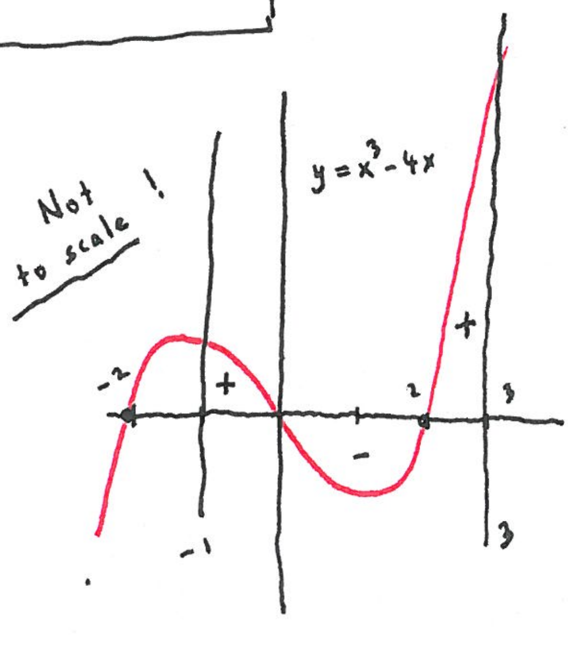
(C) $\displaystyle\int_{-1}^{3} |x^3-4x|\,dx$

$$= \int_{-1}^{0}(x^3-4x)\,dx + \int_{0}^{2}(4x-x^3)\,dx + \int_{2}^{3}(x^3-4x)\,dx$$

$$= \left.\frac{x^4}{4} - 2x^2\right|_{-1}^{0} + \left.2x^2 - \frac{x^4}{4}\right|_{0}^{2} + \left.\frac{x^4}{4} - 2x^2\right|_{2}^{3}$$

$$= -\frac{1}{4} + 2 + 8 - 4 + \frac{81}{4} - 18 - (4-8)$$

$$= \boxed{12}$$



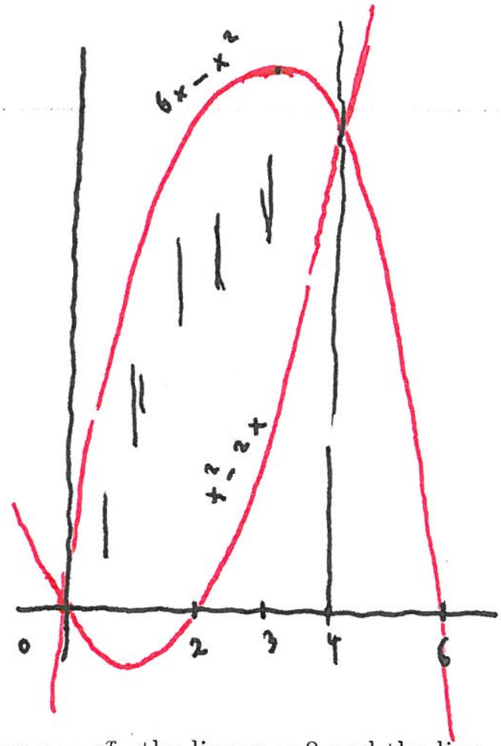
2. (10 + 10)

(A) Find the area of the finite region bounded by the curves $y = x^2 - 2x$ and $y = 6x - x^2$.

The area $= \displaystyle\int_0^4 \left[6x - x^2 - (x^2 - 2x)\right] dx$

$= \displaystyle\int_0^4 (8x - 2x^2)\, dx = 4x^2 - \frac{2}{3}x^3 \Big|_0^4$

$= 64 - \frac{2}{3}64 = \boxed{\frac{64}{3}}$

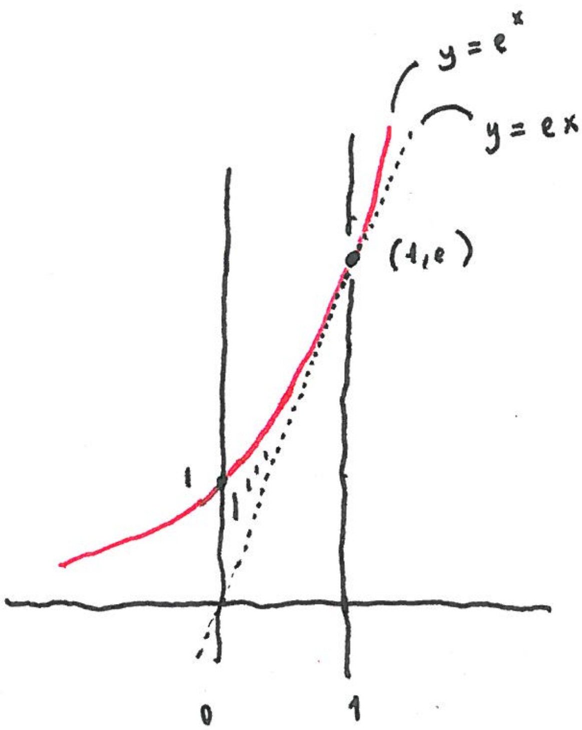


(B) Find the area of the finite region bounded by the curve $y = e^x$, the line $x = 0$ and the line tangent to $y = e^x$ at $(1, e)$.

The area $= \displaystyle\int_0^1 (e^x - ex)\, dx$

$= e^x - \frac{e}{2}x^2 \Big|_0^1 = e - \frac{e}{2} - 1$

$= \boxed{\frac{e}{2} - 1}$



3. (8 + 8 + 8) Compute the following integrals.

(A) $\int \left(\sin(x)\right)^{31/17} \left(\cos(x)\right)^3 \, dx \overset{u=\sin(x)}{=} \int u^{31/17} (1-u^2)\, du$

$= \int \left(u^{31/17} - u^{65/17}\right) du = \frac{u^{48/17}}{48/17} - \frac{u^{82/17}}{82/17}$

$= \boxed{\frac{17}{48}\left(\sin(x)\right)^{48/17} - \frac{17}{82}\left(\sin(x)\right)^{82/17}} + C$

(B) $\int \left(\ln(x)\right)^2 dx \overset{x=e^u}{=} \int u^2 e^u du \overset{\text{" By parts "}}{=} u^2 e^u - \int 2ue^u du$

$= u^2 e^u - 2\left(ue^u - \int e^u du\right) = e^u\left(u^2 - 2u + 2\right)$

$= \boxed{x\left[\left(\ln(x)\right)^2 - 2\ln(x) + 2\right]} + C$

(C) $\int x^2 \sin(3x)\, dx \overset{\text{" By parts "}}{=} x^2\left(-\frac{1}{3}\cos(3x)\right) - \int 2x\left(-\frac{1}{3}\cos(3x)\right) dx$

$= -\frac{1}{3}x^2\cos(3x) + \frac{2}{3}\left[x\left(\frac{1}{3}\sin(3x)\right) - \int\left(-\frac{1}{3}\sin(3x)\right) dx\right]$

$= \boxed{-\frac{1}{3}x^2\cos(3x) + \frac{2}{9}x\sin(3x) + \frac{2}{27}\cos(3x)} + C$

4. (9 + 7)

(A) Find the volume of the solid of revolution obtained by rotating the finite region bounded by the curves $y = x - x^3$ and $y = x - x^2$ about the x-axis.

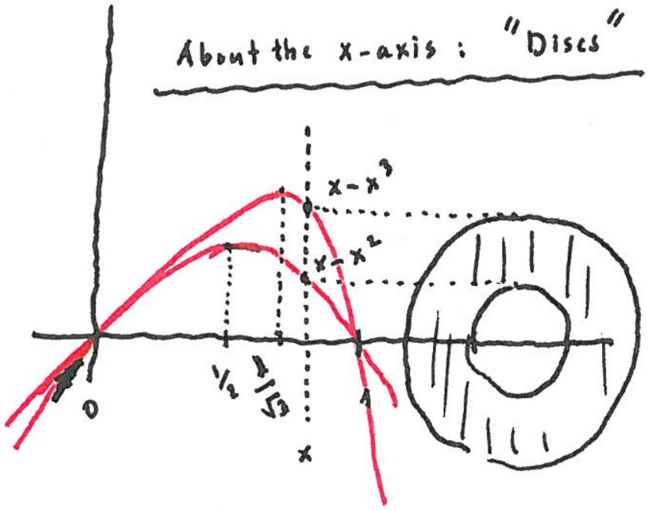
$$\text{The volume} = \int_0^1 \pi \left[(x-x^3)^2 - (x-x^2)^2 \right] dx$$

$$= \pi \int_0^1 \left[x^2 - 2x^4 + x^6 - (x^2 - 2x^3 + x^4) \right] dx$$

$$= \pi \int_0^1 (2x^3 - x^4 + x^6)\, dx$$

$$= \pi \left(\frac{2x^4}{4} - \frac{x^5}{5} + \frac{x^7}{7} \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{2} - \frac{1}{5} + \frac{1}{7} \right) = \boxed{\frac{31\pi}{70}}$$



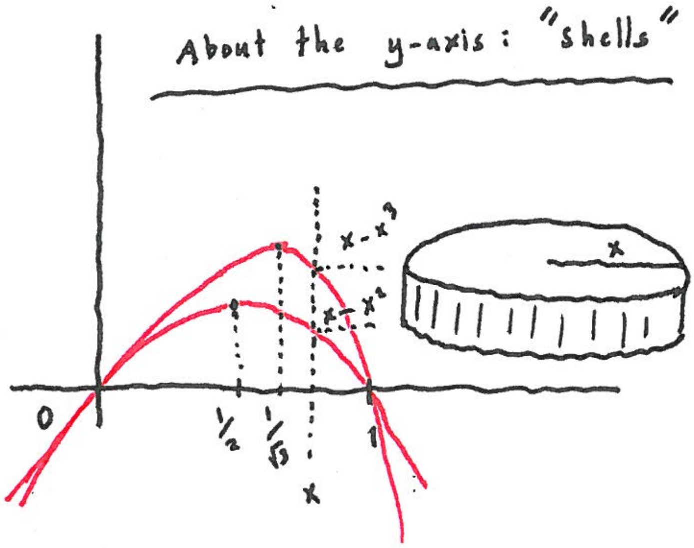
(B) Find the volume of the solid of revolution obtained by rotating the finite region bounded by the curves $y = x - x^3$ and $y = x - x^2$ about the y-axis.

$$\text{The volume} = \int_0^1 2\pi x \left[x - x^3 - (x - x^2) \right] dx$$

$$= 2\pi \int_0^1 (x^3 - x^4)\, dx$$

$$= 2\pi \left(\frac{x^4}{4} - \frac{x^5}{5} \right) \Big|_0^1$$

$$= 2\pi \left(\frac{1}{4} - \frac{1}{5} \right) = \boxed{\frac{\pi}{10}}$$



trouble!

5. (9 + 7) Decide if the following improper integrals are convergent.

(A) $\displaystyle\int_{1071}^{\infty} \frac{dx}{\sqrt{(7x+3)^3}} \overset{u=7x+3}{=} * \int_{*}^{\infty} \frac{1}{u^{3/2}}\,du$. $\boxed{\text{Convergent}}$

as $3/2 > 1$.

trouble!

(B) $\displaystyle\int_{-3/7}^{1389} \frac{\arcsin\left(\frac{x^{14}}{x^{14}+2}\right)}{\sqrt[5]{\left(7x+3\right)^3}}\,dx$

Note that $\displaystyle\int_{-3/7}^{*} \frac{1}{\sqrt[5]{(7x+3)^3}}\,dx \overset{u=7x+3}{=} * \int_{0}^{*} \frac{1}{u^{3/5}}\,du$ is convergent since $3/5 < 1$.

As $0 \le \arcsin\left(\frac{x^{14}}{x^{14}+2}\right) \le 1$ we may apply the "comparison rule" for improper integrals with <u>positive</u> integrands to conclude that the improper integral in question is $\boxed{\text{convergent.}}$