METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY FINAL EXAM	
Code : *MAT 119* Acad.Year: *2013-2014* Semester : *SUMMER* Date : *15.7.2014* Time : *09:00* Duration : *120 minutes*	Last Name: Name : KEY Department: Signature: Student No.: Section: 6 QUESTIONS ON 6 PAGES TOTAL 105 POINTS

1. (15)	2. (10)	3. (36)	4. (15)	5. (10)	6. (19)	

Show your work! Please draw a $\boxed{\text{box}}$ around your answers!

1. *(5×3 =15 pts)* Find the following limits. Show and explain your work.

(a) $\lim_{x\to 1} \frac{\ln x}{x^2-1} \overset{L'HR}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{2x} = \frac{1}{2}$

(b) $\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{\arcsin x} = \lim_{x\to 0^+} \frac{\frac{\frac{1}{x}}{e^{\frac{1}{x}}}}{\frac{\arcsin x}{x}}$

$$\left.\begin{array}{l} \lim_{x\to 0^+} \frac{\frac{1}{x}}{e^{\frac{1}{x}}} \overset{L'HR}{=} \lim_{x\to 0^+} \frac{-\frac{1}{x^2}}{e^{\frac{1}{x}}\cdot -\frac{1}{x^2}} = 0 \\ \lim_{x\to 0^+} \frac{\arcsin x}{x} \overset{L'HR}{=} \lim_{x\to 0^+} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1 \end{array}\right\} \Rightarrow \lim_{x\to 0^+} \frac{\frac{\frac{1}{x}}{e^{\frac{1}{x}}}}{\frac{\arcsin x}{x}} = \frac{0}{1} = 0$$

(c) $\lim_{x\to 0} (\tan x)^x = \lim_{x\to 0} e^{\ln(\tan x)^x}$

$$= \lim_{x\to 0} e^{x\ln(\tan x)}$$

$$= e^{\lim_{x\to 0} \frac{\ln(\tan x)}{\frac{1}{x}}}$$

$$\overset{L'HR}{=} e^{\lim_{x\to 0} \frac{\frac{1}{\tan x}\cdot \sec^2 x}{-\frac{1}{x^2}}}$$

$$= e^{\lim_{x\to 0} \frac{-x^2}{\sin x \cdot \cos x}}$$

$$= e^{\lim_{x\to 0} \frac{-x}{\sin x}\cdot\frac{x}{\cos x}} = e^0 = 1$$

2. *(5+5=10 pts)* This problem has three unrelated parts about derivative of functions.

(a) $\frac{d}{dx}\left(\frac{(x^3+1).\arctan x.\ln x}{\sqrt{x^2+1}.e^x.\tan x}\right)$ say $y = \frac{(x^3+1)\arctan x . \ln x}{\sqrt{x^2+1} . e^x . \tan x}$

$$\Rightarrow \ln y = \ln(x^3+1) + \ln(\arctan x) + \ln(\ln x) - \frac{1}{2}\ln(x^2+1) - x - \ln(\tan x)$$

by taking the derivative,

$$\frac{y'}{y} = \frac{3x^2}{x^3+1} + \frac{\frac{1}{1+x^2}}{\arctan x} + \frac{\frac{1}{x}}{\ln x} - \frac{1}{2}\frac{2x}{x^2+1} - 1 - \frac{\sec^2 x}{\tan x}$$

$$\Rightarrow y' = \left[\frac{3x^2}{x^3+1} + \frac{\frac{1}{1+x^2}}{\arctan x} + \frac{\frac{1}{x}}{\ln x} - \frac{x}{x^2+1} - 1 - \frac{\sec^2 x}{\tan x}\right] \cdot \frac{(x^3+1)\arctan x . \ln x}{\sqrt{x^2+1} . e^x . \tan x}$$

(b) $\frac{d}{dx}\left(x^{\ln(x^x)}\right) = \left(x^{x\ln x}\right)'$ say $y = x^{x\ln x}$

$$\ln y = \ln x^{x\ln x} = x\ln x . \ln x$$

by taking derivative,

$$\frac{y'}{y} = 1.(\ln x)^2 + x . 2\ln x . \frac{1}{x}$$

$$\Rightarrow y' = \left[(\ln x)^2 + 2\ln x\right] . x^{x\ln x}$$

3. *(6 × 6=36 pts)* Compute the following integrals.

(a) $\int x^5 e^{-x^3}\,dx$ say $s = x^3 \Rightarrow ds = 3x^2 dx$ then integral becomes

$$= \int s e^{-s} \cdot \frac{ds}{3} \underset{\substack{\text{use int} \\ \text{by parts} \\ u = s\,;\, dv = e^{-s} ds \\ du = ds\,;\, v = -e^{-s}}}{=\!=\!=} \frac{1}{3}\left[-se^{-s} - \int -e^{-s} ds\right] = \frac{1}{3}\left[-se^{-s} - e^{-s}\right] + C$$

$$= -\frac{1}{3} \cdot e^{-x^3}(x^3+1) + C$$

(b) $\int \frac{\sec^6\theta}{\tan^2\theta}\,d\theta$ say $s = \tan\theta \Rightarrow ds = \sec^2\theta\, d\theta$ and $s^2+1 = \sec^2\theta$

then integral becomes,

$$\int \frac{(s^2+1)^2}{s^2} ds = \int \frac{s^4+2s^2+1}{s^2} ds = \int s^2 + 2 + s^{-2} ds$$

$$= \frac{s^3}{3} + 2s - \frac{1}{s} + C$$

$$= \frac{\tan^3\theta}{3} + 2\tan\theta - \cot\theta + C$$

(c) $\int x \sin^2 x \cos x\,dx \underset{\substack{\text{use int. by parts} \\ u = x\,;\, dv = \sin^2 x \cos x\,dx \\ du = dx\,;\, v = \frac{\sin^3 x}{3}}}{=\!=\!=} \frac{x.\sin^3 x}{3} - \frac{1}{3}\int \sin^3 x\,dx$

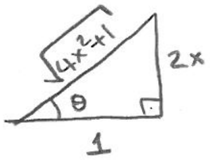
$$= \frac{x\sin^3 x}{3} - \frac{1}{3}\int (1-\cos^2 x).\sin x\,dx$$ say $s = \cos x \Rightarrow ds = -\sin x\,dx$

so, $\int (1-\cos^2 x)\sin x\,dx = \int (1-s^2).-ds = \frac{s^3}{3} - s = \frac{\cos^3 x}{3} - \cos x$. then our

integral becomes

$$= \frac{x.\sin^3 x}{3} - \frac{1}{3}\left(\frac{\cos^3 x}{3} - \cos x\right) + C$$

(d) $\int \frac{1}{x\sqrt{4x^2+1}}dx$ say $\tan\theta = 2x \Rightarrow \sec^2\theta\, d\theta = 2\,dx$



integral becomes, $= \int \frac{1}{\frac{\tan\theta}{\cancel{2}}\sqrt{\tan^2\theta+1}} \cdot \frac{\sec^2\theta\, d\theta}{\cancel{2}} = \int \frac{\sec^2\theta\, d\theta}{\tan\theta\,\sec\theta}$

$$= \int \sec\theta\,\cot\theta\, d\theta = \int \csc\theta\, d\theta$$

$$= \ln|\csc\theta - \cot\theta| + C$$

$$= \ln\left|\frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x}\right| + C$$

(e) $\int_0^1 \frac{(x-1)e^x}{x^2}dx = \lim_{s\to 0^+} \int_s^1 \frac{xe^x}{x^2} - \frac{e^x}{x^2}\, dx = \lim_{s\to 0^+}\left[\int_s^1 \frac{e^x}{x}dx - \int \frac{e^x}{x^2}dx\right]$

use int. by parts to the first integral; $u = \frac{1}{x}$; $dv = e^x dx \Rightarrow du = -\frac{dx}{x^2}$; $v = e^x$

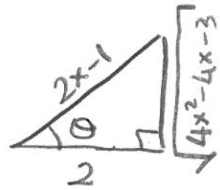
integral becomes, $= \lim_{s\to 0^+}\left[\frac{e^x}{x} + \cancel{\int \frac{e^x}{x^2}dx} - \cancel{\int \frac{e^x}{x^2}dx}\right]\Bigg|_s^1$

$$= \lim_{s\to 0^+}\left[e - \frac{e^s}{s}\right]$$

$= -\infty$ divergent

(f) $\int \sqrt{4x^2-4x-3}\,dx = \int \sqrt{(2x-1)^2-4}\,dx$ say $2\sec\theta = 2x-1$

$\Rightarrow \sec\theta\tan\theta\, d\theta = 2\,dx$



integral becomes,

$$= \int \sqrt{4\sec^2\theta - 4}\,\sec\theta\tan\theta\, d\theta = 2\int \sec\theta\,\tan^2\theta\, d\theta$$

Remember! $\int \sec\theta\tan^2\theta\, d\theta = \int \tan\theta\sec\theta\tan\theta\, d\theta$ use int. by parts

$u = \tan\theta$; $dv = \sec\theta\tan\theta\, d\theta$

$\Rightarrow du = \sec^2\theta\, d\theta$; $v = \sec\theta$.

$$= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta(\tan^2\theta - 1)\, d\theta$$

$\Rightarrow \int \sec\theta\tan^2\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta + \int \sec\theta\, d\theta$

$\Rightarrow 2\int \sec\theta\tan^2\theta\, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|$

our integral becomes, $= \frac{2x-1}{2} \cdot \frac{\sqrt{4x^2-4x-3}}{2} + \ln\left|\frac{2x-1}{2} + \frac{\sqrt{4x^2-4x-3}}{2}\right| + C$

4. *(5×3=15 pts)* Let **R** be the region between $y = \sin x$ and $y = \sin 2x$ when $0 \le x \le \pi$.

(a) Calculate the area of the region **R**.

Let's check the intersections: $\sin x = \sin 2x \Rightarrow \sin x - 2\sin x \cos x = 0$

$\Rightarrow \sin x\,(1 - 2\cos x) = 0$

$\Rightarrow x = 0\,;\ x = \frac{\pi}{3}$ solutions in $[0,\pi]$.

$$\text{Area} = \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x)\,dx + \int_{\frac{\pi}{3}}^{\pi} (\sin x - \sin 2x)\,dx$$

$$= \left(-\frac{\cos 2x}{2} + \cos x\right)\Big|_0^{\frac{\pi}{3}} + \left(-\cos x + \frac{\cos 2x}{2}\right)\Big|_{\frac{\pi}{3}}^{\pi}$$

$$= \left[\left(\frac{1}{4} + \frac{1}{2}\right) - \left(-\frac{1}{2} + 1\right)\right] + \left[\left(1 - \frac{1}{2}\right) - \left(-\frac{1}{2} - \frac{1}{4}\right)\right] = \frac{1}{4} + \frac{5}{4} = \frac{3}{2}$$

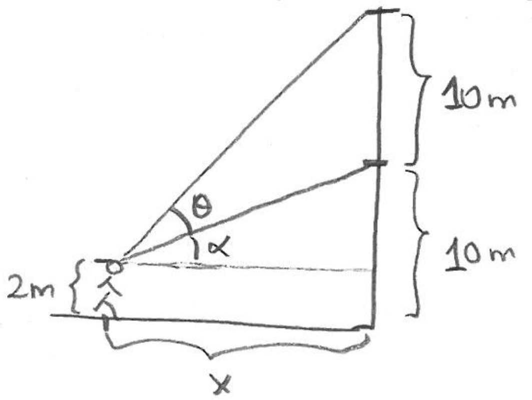
(b) Write the integral(s) which gives the volume of the solid obtained by revolving the region **R** about the line $y = -1$.

$$\text{Volume} = \int_0^{\frac{\pi}{3}} \pi\left[(\sin 2x - (-1))^2 - (\sin x - (-1))^2\right] dx + \int_{\frac{\pi}{3}}^{\pi} \pi\left[(\sin x - (-1))^2 - (\sin 2x - (-1))^2\right] dx$$

(c) Write the integral(s) which the volume of the solid obtained by revolving the region **R** about the line $x = 0$.

$$\text{Volume} = \int_0^{\frac{\pi}{3}} 2\pi x\,(\sin 2x - \sin x)\,dx + \int_{\frac{\pi}{3}}^{\pi} 2\pi x\,(\sin x - \sin 2x)\,dx$$

5. *(10 pts)* A 2m tall man wants to watch the results of the President election from a big screen on a public open space. A 10 m tall screen is located on a wall 10 m above the ground. How far would he stay from the wall to get the best view?



We need to maximize the angle θ

$$\tan\theta = \tan((\theta+\alpha)-\alpha)$$

$$= \frac{\tan(\theta+\alpha)-\tan\alpha}{1+\tan(\theta+\alpha)\cdot\tan\alpha}$$

$$= \frac{\frac{18}{x}-\frac{8}{x}}{1+\frac{18}{x}\cdot\frac{8}{x}} = \frac{\frac{10}{x}}{\frac{x^2+144}{x^2}} = \frac{10x}{x^2+144}$$

$$\theta = \tan^{-1}\left(\frac{10x}{x^2+144}\right)$$

$$\Rightarrow \theta' = \frac{1}{1+\left(\frac{10x}{x^2+144}\right)^2}\cdot\left(\frac{10\cdot(x^2+144)-10x\cdot 2x}{(x^2+144)^2}\right)$$

$$\Rightarrow \theta' = \frac{1440-10x^2}{(x^2+144)^2+(10x)^2} = 0 \Longrightarrow x^2 = 144 \Rightarrow x = 12$$

$(\cancel{x=-12})$ nonsense!

to find critical points (denominator is always positive!)

x	0		12	
θ'		+	0	−

So the local max is actually global max.

He should stay 12 m away from the wall.

6. *(3+2+2+4+4+4=19 pts)* Let $f(x) = (x+1)e^{-x}$. By following these steps sketch the graph of $f(x)$.

(a) Find the domain, x-intercepts and y-intercept of $f(x)$.

$D_f : \mathbb{R}$; $(x+1)e^{-x} = 0 \Rightarrow x_{int} = -1$; $y_{int} = (0+1)e^{-0} = 1$

(b) Find ALL of the asymptotes of $f(x)$.

Since, the $D_f = \mathbb{R}$, there is no vertical asymptote

$\lim\limits_{x \to -\infty} (x+1)e^{-x} = -\infty$; $\lim\limits_{x \to \infty} (x+1)e^{-x} \underset{=}{\overset{LHR}{=}} \lim\limits_{x \to \infty} \frac{1}{e^{x}} = 0$.

$y=0$ is the horizontal asymptote.

(c) Check the symmetry and periodicity.

$(-x+1)e^{x} \neq \mp (x+1)e^{-x}$ No symmetry.

Since there is no trigonometric function, it is not periodic.

(d) Find the intervals of increase/decrease and local max/min points of $f(x)$.

$f'(x) = 1.e^{-x} + (x+1).-e^{-x} = -xe^{-x} = 0 \Rightarrow x = 0$

to find critical points

$f(x)$ is increasing on : $(-\infty, 0)$

$f(x)$ is decreasing on : $(0, \infty)$

$f(x)$ has local max at $x=0$.

local max

x		0		1	
$f'(x)$	$+$	0	$-$		$-$
$f''(x)$	$-$		$-$	0	$+$

inflection points

(e) Find the intervals of concavity and inflection points of $f(x)$.

$f''(x) = -1.e^{-x} + (-x).-e^{-x} = (x-1)e^{-x} = 0 \Rightarrow x = 1$

to find inflection points.

$f(x)$ is concave up on : $(1, \infty)$

$f(x)$ is concave down on : $(-\infty, 1)$

(f) Sketch the graph of $f(x)$.