METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY MIDTERM 1	
Code : *MAT 119* Acad.Year: *2013-2014* Semester : *Spring* Date : *05.04.2014* Time : *9:40* Duration : *120 min*	Last Name: Name : Student No.: Department: Section: Signature : 7 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1. (16)	2. (18)	3. (10)	4. (12)	5. (12)	6. (12)	7. (20)	Bonus		

1. *(4×4=16 pts)* Find the following limits, if they exist. Show your work.

(a) $\lim_{x\to 0} \frac{x}{\sqrt{4+x}-\sqrt{4-x}} = \lim_{x\to 0} \frac{x\cdot(\sqrt{4+x}+\sqrt{4-x})}{(\sqrt{4+x}-\sqrt{4-x})(\sqrt{4+x}+\sqrt{4-x})}$

$= \lim_{x\to 0} \frac{x\cdot(\sqrt{4+x}+\sqrt{4-x})}{(4+x)-(4-x)} = \lim_{x\to 0} \frac{\cancel{x}(\sqrt{4+x}+\sqrt{4-x})}{2\cancel{x}} \underset{\text{D.S.P.}}{=} \frac{2+2}{2} = 2.$

(b) $\lim_{x\to 0} \frac{\sin(\sin(3x))}{4x} = \lim_{x\to 0} \frac{\sin(\sin(3x))}{\sin(3x)}\cdot\frac{\sin(3x)}{4x}\cdot\frac{3}{3}$

$= \lim_{x\to 0} \underbrace{\frac{\sin(\sin(3x))}{\sin(3x)}}_{\to 1}\cdot\underbrace{\frac{\sin(3x)}{3x}}_{\to 1}\cdot\frac{3}{4} = \frac{3}{4}$

as $x\to 0$

(c) $\lim_{x\to\infty} \frac{|\sin x|}{x}$

$0 \leq |\sin x| \leq 1$

$0 \leq \frac{|\sin x|}{x} \leq \frac{1}{x}$

$\to 0$

by Squeeze Theorem, limit is 0.

(d) $\lim_{x\to -\infty} \frac{\sqrt{x^2+2x}-\sqrt{4x^2+3x}}{3x+4} = \lim_{x\to -\infty} \frac{\sqrt{x^2(1+\frac{2}{x})}-\sqrt{4x^2(1+\frac{3}{4x})}}{3x(1+\frac{4}{3x})}$

$= \lim_{x\to -\infty} \frac{|x|\cdot\sqrt{1+\frac{2}{x}}-|2x|\sqrt{1+\frac{3}{4x}}}{3x(1+\frac{4}{3x})} = \lim_{x\to -\infty} \frac{-x\sqrt{1+\frac{2}{x}}+2x\sqrt{1+\frac{3}{4x}}}{3x(1+\frac{4}{3x})}$

$= \lim_{x\to -\infty} \frac{\cancel{-x}(\sqrt{1+\frac{2}{x}}-2\sqrt{1+\frac{3}{4x}})}{3\cancel{x}(1+\frac{4}{3x})} = \frac{1}{3}$

$\lim_{x\to -\infty} \frac{1}{x^r} = 0$

$r>0$ and $r\in\mathbb{Q}$

2. *(3×6=18 pts)* This problem has three unrelated parts.

(a) $f(x) = \begin{cases} -2x^2 + 3x + 4, & \text{if } x \leqslant 1 \\ -x + 3, & \text{if } x > 1 \end{cases}$. Is $f(x)$ differentiable at $x = 1$.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} -x + 3 \underset{\text{D.S.P.}}{=} -1 + 3 = 2$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} -2x^2 + 3x + 4 \underset{\text{D.S.P.}}{=} -2 + 3 + 4 = 5$$

$f(x)$ is not continuous at $x = 1$, so it can't be differentiable.

(b) $g(x) = \dfrac{(x^2 + 1)\cos x}{\tan(x + 1)}$. Calculate $g'(x)$.

$$g'(x) = \frac{(2x \cdot \cos x - (x^2+1) \cdot \sin x)\tan(x+1) - (x^2+1)\cos x \cdot \sec^2(x+1) \cdot 1}{\tan^2(x+1)}$$

(c) $h(x) = \sqrt{\sec x + \sin(x^4 + 1)}$. Calculate $h'(x)$.

$$h(x) = (\sec x + \sin(x^4 + 1))^{1/2}$$

$$h'(x) = \frac{1}{2}(\sec x + \sin(x^4+1))^{\frac{-1}{2}} \cdot (\sec x \cdot \tan x + \cos(x^4+1) \cdot 4x^3)$$

3. *(10 pts)* Find an equation of the tangent line to the curve given by the equation

$$x\sin(xy - y^2) = x^2 - 1$$

at (1,1).

$$\frac{d}{dx}\left(x \cdot \sin(xy - y^2)\right) = \frac{d}{dx}\left(x^2 - 1\right)$$

$$\sin(xy-y^2) + x\cdot\cos(xy-y^2)\cdot\left(y + x\cdot\frac{dy}{dx} - 2y\cdot\frac{dy}{dx}\right) = 2x$$

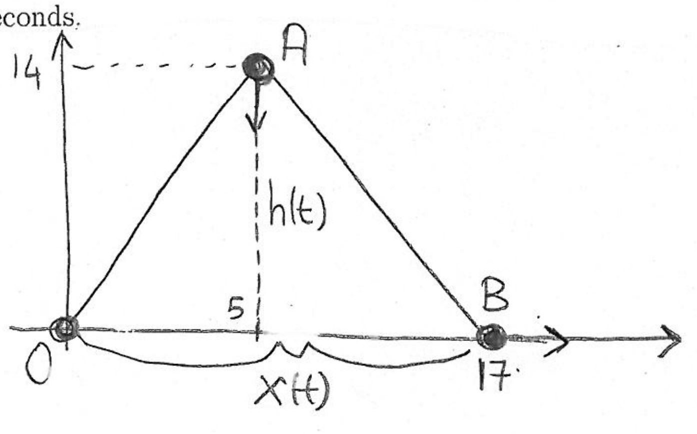
$$\left(x\cdot\cos(xy-y^2)\cdot(x-2y)\right)\frac{dy}{dx} = 2x - \sin(xy-y^2) - xy\cos(xy-y^2)$$

$$\frac{dy}{dx} = \frac{2x - \sin(xy-y^2) - xy\cos(xy-y^2)}{x\cdot\cos(xy-y^2)\cdot(x-2y)}$$

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{2\cdot 1 - \sin(0) - 1\cdot\cos(0)}{1\cdot\cos(0)\cdot(1-2)} = \frac{1}{-1} = -1$$

$$y - 1 = -1(x-1) \Rightarrow \boxed{y = -x + 2}$$

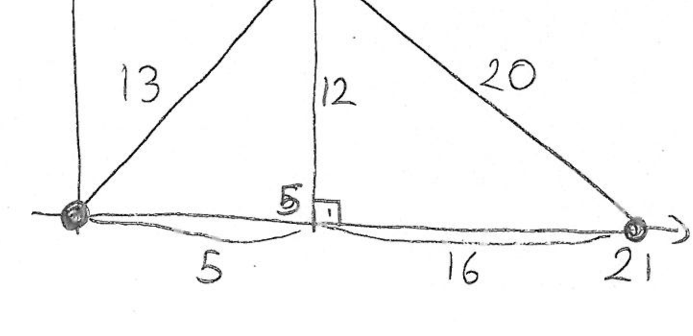
4. *(12 pts)* Vertices of a triangle are at **O**(0,0), **A**(5,14) and **B**(17,0). Suppose the vertex **A** starts moving down with a rate of 1 unit/s and at the same time, the vertex **B** starts moving to the right with a rate of 2 unit/s. Find the rate of change of the area of the triangle after 2 seconds.



$$A = \frac{x\cdot h}{2}$$

$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{dx}{dt}\cdot h + x\cdot\frac{dh}{dt}\right)$$

After 2 seconds we will have the following triangle



$$\left.\frac{dA}{dt}\right|_{t=2} = \frac{1}{2}\left(2\cdot 12 + 21\cdot(-1)\right)$$

$$= \frac{3}{2}\ \text{unit}^2/\text{s}.$$

5. *(12 pts)* Find the alsolute maximum and absolute minimum of the function $f(x) = \sin^2 x + \cos x$ on $[0, \frac{\pi}{2}]$.

f is a continuous function on a closed interval $[0, \pi/2]$, hence by Extreme Value Theorem, the absolute max. and min. exist. By the closed interval method, they can be at critical points or boundary points

Critical Points

$$f'(x) = 2\sin x \cdot \cos x - \sin x = 2\sin x\left(\cos x - \frac{1}{2}\right)$$

$$f'(x) = 0 \Rightarrow \sin x = 0 \quad \text{or} \quad \cos x = \frac{1}{2}$$

$$x = 0 \qquad x = \frac{\pi}{3}$$

Boundary point.

$$f\left(\frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{3}{4} + \frac{2}{4} = \frac{5}{4}$$

Global max. value

Boundary Points

$f(0) = 1$

$f(\pi/2) = 1$

Global min. value.

6. *(2× 5=10pts)* Two cars were making a drift race. Both cars started the race at the same time and they finished the race at the same time. Determine whether the given statement is True or False.

(T/F) Both cars should have the same instantaneous velocity all the time.

(T/F) Both cars should have the same average velocity at the end of the time.

(T/F) Both cars should have the same instantaneous velocity at least one time.

(T/F) Both cars should have the same instantaneous velocity at most one time.

(T/F) Both cars should meet at least two times during the race except start and finish points.

7. (4+4+5+5+4=22pts) Let $f(x) = x^{\frac{2}{3}}(3-x)^{\frac{1}{3}}$. In this question we work towards sketching the graph of $f(x)$.

(a) Find the domain, x-intercepts and y-intercept of $f(x)$.

$\mathrm{Dom}(f) = \mathbb{R}$, x-int: $f(x) = 0 \Rightarrow x = 0$ or $x = 3$

y-int: $f(0) = 0 \Rightarrow y = 0$.

(b) Find the asymptotes of $f(x)$.

No Vertical Asymptotes since domain of f is $\mathbb{R}$.

$\lim_{x\to\infty} x^{2/3}(3-x)^{1/3} = -\infty$, $\lim_{x\to-\infty} x^{2/3}(3-x)^{1/3} = +\infty$

(c) Find the intervals of increase/decrease and local max/min points of $f(x)$.

$$f'(x) = \frac{2}{3}x^{-1/3}(3-x)^{1/3} + x^{2/3}\frac{1}{3}(3-x)^{-2/3}\cdot(-1) = \frac{2(3-x)^{1/3}}{3x^{1/3}} - \frac{x^{2/3}}{3\cdot(3-x)^{2/3}}$$

$$= \frac{2(3-x) - x}{3x^{1/3}\cdot(3-x)^{2/3}} = \frac{(6-3x)}{3\cdot x^{1/3}(3-x)^{2/3}}$$

Critical points are 0, 2, 3

0 is a local min. pt.

2 is a local max. pt.

		0		2		3	
f'	$-$		$+$		$-$		$-$
	↘		↗		↘		↘

(d) Find the intervals of concavity and inflection points of $f(x)$.

$$f''(x) = \frac{-3\cdot 3x^{1/3}(3-x)^{2/3} - (6-3x)\left(\cancel{3}\cdot\frac{1}{\cancel{3}}x^{-2/3}\cdot(3-x)^{2/3} + \cancel{3}x^{1/3}\cdot\frac{2}{\cancel{3}}(3-x)^{-1/3}\cdot(-1)\right)}{9\cdot x^{2/3}(3-x)^{4/3}}$$

$$= \frac{-9\cdot x^{1/3}(3-x)^{2/3} - (6-3x)\left(\frac{(3-x)^{2/3}}{x^{2/3}} - \frac{2x^{1/3}}{(3-x)^{1/3}}\right)}{9\cdot x^{2/3}(3-x)^{4/3}}$$

$$= \frac{-9\cdot x\cdot(3-x) - (6-3x)(3-x-2x)}{9\cdot x^{4/3}\cdot(3-x)^{5/3}}$$

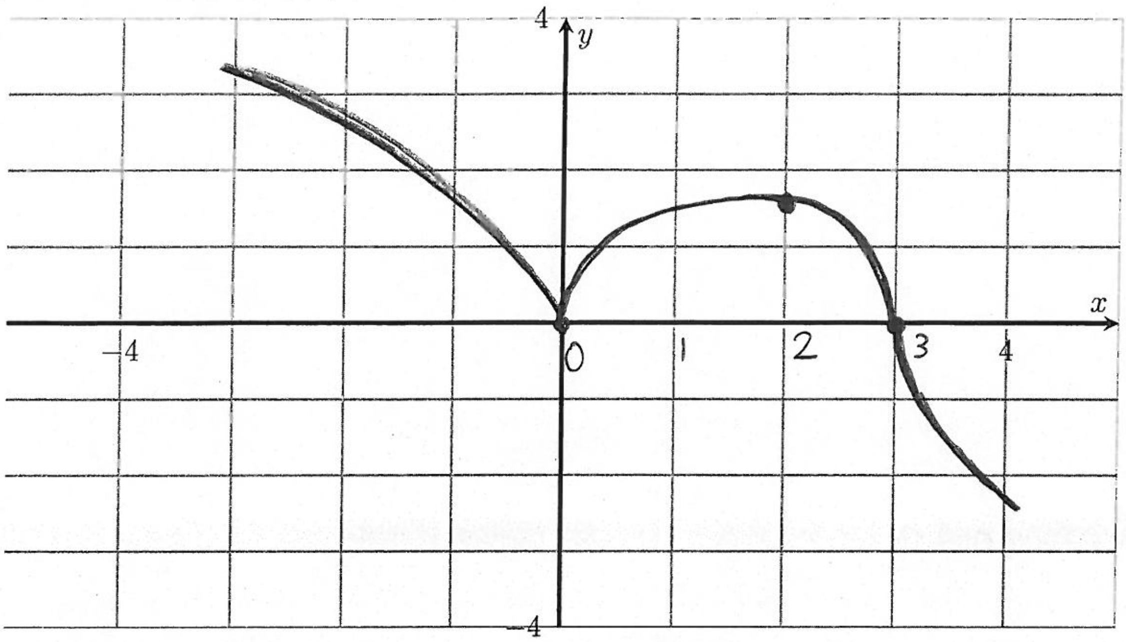
$$= \frac{(-\cancel{27x} + 9x^2) - (18 - \cancel{27x} + 9x^2)}{9\cdot x^{4/3}\cdot(3-x)^{5/3}} = \frac{-18}{9x^{4/3}\cdot(3-x)^{5/3}}$$

Possible inflection points are 0 or 3

$x = 3$ is an inflection point.

		0		3	
f''	$-$		$-$		$+$

(**e**) Sketch the graph of $f(x)$.



Bonus *(5pts)*(No partial points will be given.) Show that $\lim_{x \to 1} \dfrac{x+2}{2x-3} = -3$ by using the precise definition of limit.

For any $\varepsilon > 0$ given, choose $\delta = \min\{\frac{1}{4}, \frac{\varepsilon}{14}\}$ such that if $0 < |x-1| < \delta$, then $\left|\frac{x+2}{2x-3} + 3\right| < \varepsilon$

$$\left|\frac{x+2}{2x-3} + 3\right| = \left|\frac{7x-7}{2x-3}\right| = \frac{7}{|2x-3|} \cdot |x-1| \underset{\delta < 1/4}{<} 14 \cdot |x-1| < \varepsilon$$

Choose $\delta < \frac{1}{4}$

$$|x-1| < \frac{1}{4}$$
$$-\frac{1}{4} < x-1 < \frac{1}{4}$$
$$\frac{3}{4} < x < \frac{5}{4}$$
$$\frac{6}{4} < 2x < \frac{10}{4}$$

$$-\frac{6}{4} < 2x-3 < -\frac{2}{4}$$
$$-2 < \frac{1}{2x-3} < -\frac{4}{6}$$
$$7 \cdot \frac{4}{6} < \frac{7}{|2x-3|} < 7 \cdot 2$$