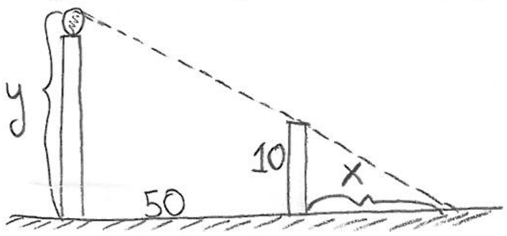
METU - NCC

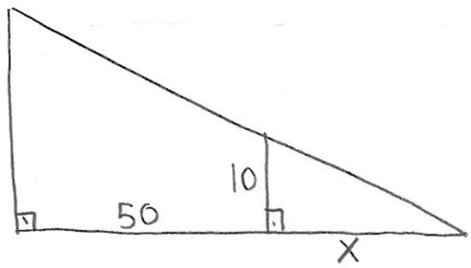
CALCULUS with ANALYTIC GEOMETRY MIDTERM 1	
Code : *MAT 119* Acad.Year: *2014-2015* Semester : *FALL* Date : *08.11.2014* Time : *9:40* Duration : *120 min*	Last Name: KEY List #: Name : Student # : Signature : 7 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1. (15)	2. (20)	3. (16)	4. (10)	5. (12)	6. (15)	7. (14)		

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(15pts)* A candle and a vertical rod of length 10cm are standing 50cm apart on a flat surface. If the candle is getting smaller with a rate of 1cm per hour, how fast is the shadow of the vertical rod increasing when the flame of the candle is 11cm high? [Note that the shadow of the vertical rod is on the ground.]





$y(t)$ = height of candle at time t.
$x(t)$ = length of shadow of rod at time t.

$$\frac{x}{x+50} = \frac{10}{y} \Rightarrow x \cdot y = 10x + 500$$

We get $\frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt} = 10 \cdot \frac{dx}{dt}$

$\frac{dy}{dt} = -1 \text{cm/h}$ When $y = 11$, $\frac{x}{x+50} = \frac{10}{11} \Rightarrow x = 500 \text{cm}$

Hence, we get $\frac{dx}{dt}(y-10) = -x \cdot \frac{dy}{dt}$

When $y=11$ $\frac{dx}{dt}(11-10) = -500 \cdot (-1)$

$$\frac{dx}{dt} = 500 \text{ cm/h}$$

2. *(4+6+4+6=20pts)* This problem has unrelated parts about limit of functions. Show your work.

(a) Compute $\lim_{x\to 0} \frac{\sin(\cos x)}{\cos x}$

$$\lim_{x\to 0} \frac{\sin(\cos x)}{\cos x} \underset{\substack{\text{by continuity} \\ \text{at } 0}}{=} \frac{\sin(\cos 0)}{\cos 0} = \frac{\sin(1)}{1} = \sin(1)$$

(b) Compute $\lim_{x\to 2} \frac{|x^2-4|}{\sqrt[3]{x-2}}$

$$\lim_{x\to 2^+} \frac{x^2-4}{\sqrt[3]{x-2}} = \lim_{x\to 2^+} \frac{(x-2).(x+2)}{(x-2)^{1/3}} = \lim_{x\to 2^+} (x-2)^{2/3}(x+2) \underset{\text{D.S.P.}}{=} 0$$

$$\lim_{x\to 2^-} \frac{-(x^2-4)}{(x-2)^{1/3}} = \lim_{x\to 2^-} \frac{-(x-2)(x+2)}{(x-2)^{1/3}} = \lim_{x\to 2^-} -(x-2)^{2/3}(x+2) \underset{\text{D.S.P.}}{=} 0$$

(c) Compute $\lim_{x\to 0}(x^2-x^3)\sin(\frac{1}{x})$

$$-1 < \sin\left(\frac{1}{x}\right) \leq 1$$

When $-1<x<1$, $x^2>x^3$, hence $x^2-x^3>0$

$$-(x^2-x^3) \leq (x^2-x^3).\sin\left(\frac{1}{x}\right) \leq (x^2-x^3)$$

by Squeeze

$$0$$

(d) Find conditions on a and b so that $\lim_{x\to\infty}[\sqrt{x^2+2x+1}-(ax+b)]$ exists, i.e. is finite.

$$\lim_{x\to\infty} [\sqrt{x^2+2x+1} - (ax+b) = \lim_{x\to\infty} \sqrt{(x+1)^2} - (ax+b)$$

$$= \lim_{x\to\infty} |x+1| - (ax+b) \underset{\substack{\text{as } x\to\infty \\ x+1>0}}{=} \lim_{x\to\infty} (1-a)x + (1-b)$$

limit is finite if and only if $(1-a)=0$ i.e. $a=1$

Name: ID:

3. *(5+5+6=16pts)* Calculate the following derivatives. Do not simplify your answers.

(a) $\frac{d}{dx}\left(\frac{(-10x+1)\sin x}{\tan(5x)}\right)$

$$= \frac{(-10\cdot\sin x + (-10x+1)\cdot\cos x)\cdot\tan(5x) - (-10x+1)\cdot\sin x\cdot\sec^2(5x)\cdot 5}{\tan^2(5x)}$$

(b) $\frac{d}{dx}\left(\sqrt[3]{x\cos(\sqrt{x+1})}\right)$

$$= \frac{1}{3}\left(x\cos(\sqrt{x+1})\right)^{-2/3}\cdot\left(1\cdot\cos(\sqrt{x+1}) + x\cdot\left(-\sin(\sqrt{x+1})\cdot\frac{1}{2}\cdot(x+1)^{-1/2}\cdot 1\right)\right)$$

(c) Find y' at the point $(1,1)$ on the curve given by the equation $(x^2+y^2-1)^3 - x^2y^3 = 0$.

$$\frac{d}{dx}\left((x^2+y^2-1)^3 - x^2y^3\right) = \frac{d}{dx}(0)$$

$$3\cdot(x^2+y^2-1)^2\cdot(2x+2y\cdot y') - 2xy^3 - 3x^2y^2\cdot y' = 0$$

$$y'\left(6\cdot(x^2+y^2-1)^2\cdot y - 3x^2y^2\right) = 2xy^3 - 6x(x^2+y^2-1)^2$$

$$y'\big|_{(1,1)} = \frac{2\cdot1\cdot1^3 - 6\cdot1\cdot(1^2+1^2-1)^2}{6(1^2+1^2-1)^2\cdot1 - 3\cdot1^2\cdot1^2} = \frac{-4}{3}$$

4. *(10)pts* Approximate $\sqrt[4]{626}$ using linearization (tangent line approximation).

$f(x) = x^{1/4} \quad f'(x) = \frac{1}{4} x^{-3/4} = \frac{1}{4\sqrt[4]{x^3}}$

Choose $a = 625$, $f(625) = 5$, $f'(625) = \frac{1}{4 \cdot 125} = \frac{1}{500}$.

$L_{625}(x) = f'(625) \cdot (x - 625) + f(625)$

$= \frac{1}{500} \cdot (x - 625) + 5$

$f(626) \approx L_{625}(626) = \frac{1}{500} \cdot (626 - 625) + 5 = 5 + \frac{1}{500}$

nearby 625

$= \frac{2501}{500}$

5 *(1.5×8=12pts)* Determine whether the statements below are true or false. Give your answer by writing **TRUE** or **FALSE**.

FALSE **(a)** If $f(x)$ and $g(x)$ are continuous at $x = a$ then $f(g(x))$ is also continuous at $x = a$.

FALSE **(b)** If $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$ then $f(x)$ is differentiable at $x = a$.

TRUE **(c)** If $f(x)$ is differentiable at $x = a$ then $\lim_{x\to a} f(x) = f(a)$.

FALSE **(d)** Suppose that $f(x)$ is differentiable everywhere. If $f(0) = 0$, $f(5) = 10$ then $f'(x) \geqslant 2$ for all $x \in (0, 5)$.

TRUE **(e)** Suppose that $f(x)$ is differentiable and the equation $f'(x) = 0$ has exactly one solution. Then for any number a the equation $f(x) = a$ has at most two solutions.

TRUE **(f)** The function $f(x) = |x^2 - 1|$ is continuous everywhere.

FALSE **(g)** If $|f(x)|$ is differentiable everywhere, then $f(x)$ is differentiable everywhere, too.

TRUE **(h)** If the Mean Value Theorem holds for $f(x)$ on $[a, b]$, then the Intermediate Value Theorem also holds.

6. *(15pts)* Find the absolute maximum and minimum of the following function $f(x)$ on $[-4, 2]$.

$$f(x) = \begin{cases} -x^2 - 4x + 5, & \text{if } x \leqslant 0 \\ |\, 4x - 4 \,| + 1, & \text{if } x > 0 \end{cases} = \begin{cases} -x^2-4x+5 & x \leq 0 \\ -4x+5 & 0<x<1 \\ 4x-3 & 1 \leq x \end{cases}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -x^2-4x+5 \underset{\text{D.S.P.}}{=} 5 \qquad f(0)=5.$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} -4x+5 \underset{\text{D.S.P.}}{=} 5$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} -4x+5 \underset{\text{D.S.P}}{=} 1 \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 4x-3 = 1, \ f(1)=1$$

So, it is continuous and Extreme Value Theorem guarantees abs max and abs. min

$$\left.\begin{aligned} &\lim_{x\to 0^-} \frac{-x^2-4x+5-5}{x-0} = \lim_{x\to 0^-} \frac{-x^2-4x}{x} = \lim_{x\to 0^-} -x-4 = -4 \\ &\lim_{x\to 0^+} \frac{|4x-4|+1-5}{x-0} = \lim_{x\to 0^+} \frac{-(4x-4)+1-5}{x} = \lim_{x\to 0^+} \frac{-4x}{x} = -4 \end{aligned}\right\} f'(0) = -4$$

$$\left.\begin{aligned} &\lim_{x\to 1^-} \frac{|4x-4|+1-1}{x-1} = \lim_{x\to 1^-} \frac{-(4x-4)+1-1}{x-1} = \lim_{x\to 1^-} \frac{-4(x-1)}{x-1} = -4 \\ &\lim_{x\to 1^+} \frac{|4x-4|+1-1}{x-1} = \lim_{x\to 1^+} \frac{(4x-4)+1-1}{x-1} = \lim_{x\to 1^+} \frac{4(x-1)}{x-1} = 4 \end{aligned}\right\} f'(1) = \text{DNE}$$

$$\left.\begin{aligned} &\text{When } x<0,\ f'(x) = -2x-4 \Rightarrow f'(-2)=0 \\ &\text{when } 0<x<1,\ f'(x) = -4\ ; \ \text{when } x>1,\ f'(x)=4 \end{aligned}\right\} \text{Critical points are } -2, 1.$$

Now let's make a list;

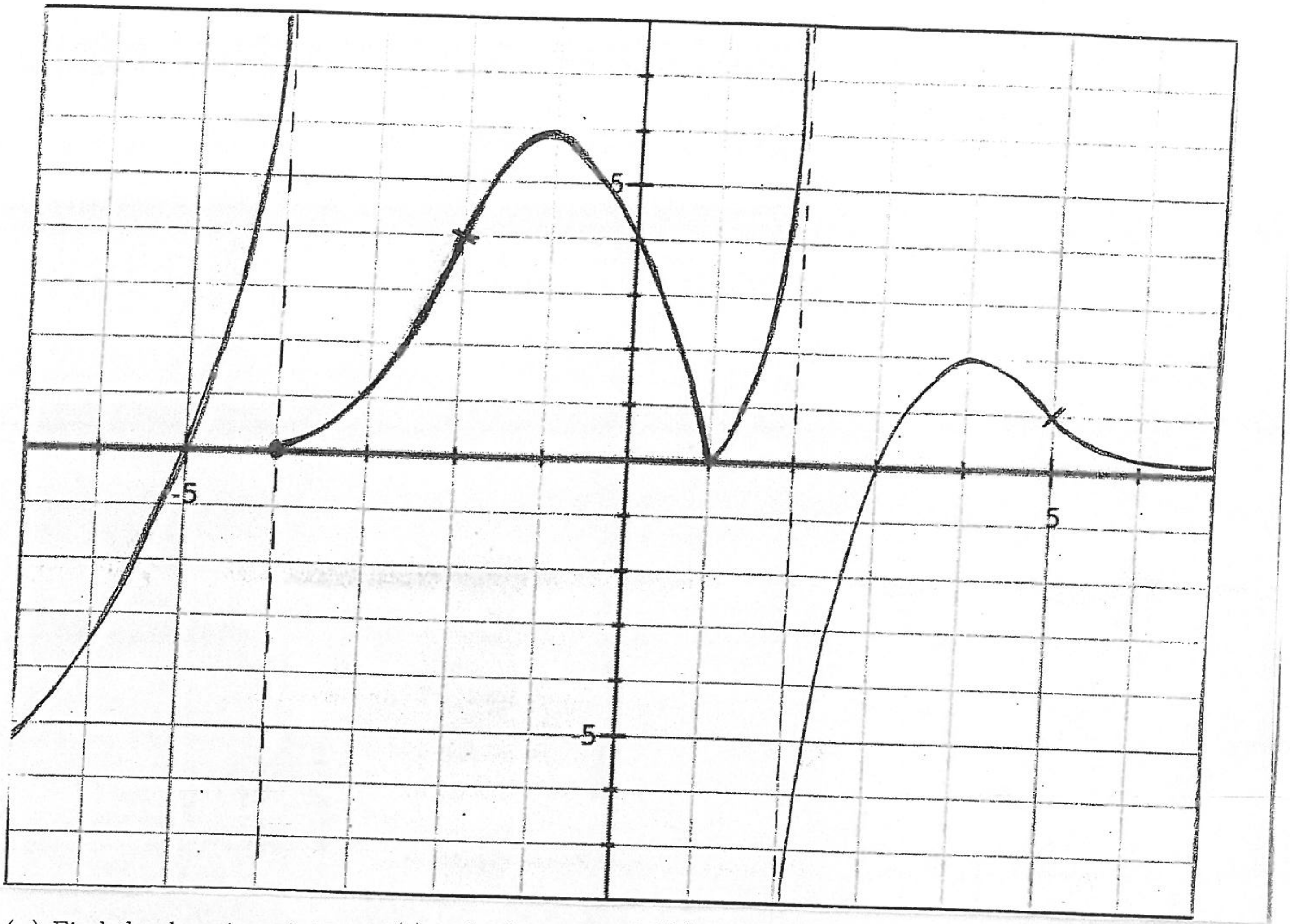
Critical points $\begin{cases} f(-2) = 9 \\ f(1) = 1 \end{cases}$

end points $\begin{cases} f(-4) = 5 \\ f(2) = 5 \end{cases}$

$f(x)$ has absolute max at $x = -2$

$f(x)$ has absolute min at $x = 1$

7. *(2+2+2+4+4=14pts)* The graph of $f(x)$ is given below. Using this graph answer the following questions.



(a) Find the domain, x-intercept(s) and y-intercept of $f(x)$.

$Dom(f) = \mathbb{R} \setminus \{2\}$. x-intercepts: $-5, -4, 1, 3$

y-intercept: 4

(b) Find the asymptotes of $f(x)$.

Horizontal Asymptote: $\lim_{x\to\infty} f(x) = 0$ so, $y = 0$

Vertical Asymptotes: $\lim_{x\to -4^-} f(x) = +\infty$, $\lim_{x\to 2^-} f(x) = +\infty$, so $x = -4, x = 2$.

(c) Is the function $f(x)$ symmetric or periodic?

There is no symmetry with respect to y-axis or origin.

It is not periodic.

(d) Find the intervals of increase/decrease and local max/min points of $f(x)$.

On $(-\infty,-4) \cup (-4,-2) \cup (1,2) \cup (2,4)$ f(x) is increasing

On $(-1,1) \cup (4,+\infty)$ f(x) is decreasing

f(-1) and f(4) are local maximum values, f(1) is a local minimum value.

(e) Find the intervals of concavity and inflection points of $f(x)$.

On $(-\infty,-4) \cup (-4,-2) \cup (1,2) \cup (5,+\infty)$, f(x) is concave up

On $(-2,1) \cup (2,5)$, f(x) is concave down

$x = -2, 1, 5$ are inflection points.