METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY MIDTERM 1	
Code : *MAT 119* Acad.Year: *2012-2013* Semester : *SUMMER* Date : *22.7.2012* Time : *17:40* Duration : *120 minutes*	Last Name: Name : Student No.: Department: Section: Signature:
	6 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1. (15)	2. (20)	3. (15)	4. (20)	5. (15)	6. (15)	Bonus		

Show your work! Please draw a $\boxed{\text{box}}$ around your answers!

1. *(3× 5pts)* Compute the following limits. Do NOT use L'Hospital Rule!

(a) $\displaystyle\lim_{x\to\infty}\frac{(\sin\pi x)^2}{x^2+1}$

$-1 \leq \sin\pi x \leq 1 \Rightarrow 0 \leq (\sin\pi x)^2 \leq 1$

$\Rightarrow 0 \leq \dfrac{(\sin\pi x)^2}{x^2+1} \leq \dfrac{1}{x^2+1}$

$\displaystyle\lim_{x\to\infty} 0 = \lim_{x\to\infty}\frac{1}{x^2+1} = 0$ Using Squeeze Theorem;

$\displaystyle\lim_{x\to\infty}\frac{(\sin\pi x)^2}{x^2+1} = 0.$

(b) $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^2+x+1}-x}{x+1} = \lim_{x\to-\infty}\frac{|x|\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}-x}{x+1} = \lim_{x\to-\infty}\frac{-x\left[\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1\right]}{x\left(1+\frac{1}{x}\right)}$

$= \dfrac{-2}{1} = -2$

(c) $\displaystyle\lim_{x\to0}\frac{1-\cos 3x}{1-\cos 5x} = \lim_{x\to0}\frac{1-\cos3x}{1-\cos5x}\cdot\frac{1+\cos3x}{1+\cos3x}\cdot\frac{1+\cos5x}{1+\cos5x}$

$\displaystyle = \lim_{x\to0}\frac{\sin^2 3x}{\sin^2 5x}\cdot\frac{1+\cos5x}{1+\cos3x}$

$\displaystyle = \lim_{x\to0}\frac{\frac{\sin^2 3x}{(3x)^2}(3x)^2}{\frac{\sin^2 5x}{(5x)^2}(5x)^2}\cdot\frac{1+\cos5x}{1+\cos3x}$

$= \dfrac{9}{25}$

2. *(4× 5pts)* Evaluate the following derivatives and integrals.

(a) $\dfrac{d}{dx}\left[\dfrac{(2x-\pi)\sin\frac{x}{2}}{(3x+\pi)\cos 4x}\right]$

$$= \frac{\left[2.\operatorname{Sin}\frac{x}{2} + (2x-\pi).\operatorname{Cos}\frac{x}{2}.\frac{1}{2}\right].(3x+\pi)\operatorname{Cos}4x - \left[3.\operatorname{Cos}4x + -\operatorname{Sin}4x.4.(3x+\pi)\right].(2x-\pi)\operatorname{Sin}\frac{x}{2}}{\left[(3x+\pi)\operatorname{Cos}4x\right]^2}$$

(b) $\dfrac{d}{dx}\left[\displaystyle\int_{2x}^{x^2}\sqrt{t^3+3t+1}\,dt\right] \overset{FTC}{=} \sqrt{x^6+3x^2+1}\,.2x - \sqrt{8x^3+6x+1}\,.2$

(c) $\displaystyle\int \frac{8x^7}{\sqrt{x^4+1}}\,dx = \int \frac{2(s-1)\,ds}{\sqrt{s}} = \int 2\sqrt{s} - \frac{2}{\sqrt{s}}\,ds = 2.\frac{s^{3/2}}{3/2} - 2.\frac{s^{1/2}}{1/2} + C$

say $x^4+1 = s$

$x^4 = s-1$

$4x^3dx = ds$

$$= \frac{4}{3}(x^4+1)^{3/2} - 4(x^4+1)^{1/2} + C$$

(d) $\displaystyle\int_0^{\pi}\frac{d}{dx}\left[\frac{(2x-\pi)\sin\frac{x}{2}}{(3x+\pi)\cos 4x}\right]dx \overset{FTC}{=} \frac{(2\pi-\pi)\operatorname{Sin}\frac{\pi}{2}}{(3\pi+\pi)\operatorname{Cos}4\pi} - \frac{(0-\pi)\operatorname{Sin}0}{(0+\pi)\operatorname{Cos}0} = \frac{1}{4}$

3. *(15 pts)* Find absolute maximum and minimum of the function $f(x) = (x+5)^{2/3}(4-x)^{1/3}$ on $[-4, 7]$.

$$f'(x) = \frac{2}{3}\frac{(4-x)^{1/3}}{(x+5)^{1/3}} - \frac{1}{3}\frac{(x+5)^{2/3}}{(4-x)^{2/3}}$$

$$= \frac{2(4-x) - (x+5)}{3(x+5)^{1/3}(4-x)^{2/3}}$$

$$= \frac{3-3x}{3(x+5)^{1/3}(4-x)^{2/3}}$$

So, the critical points are;

$x = 1$ makes f' 0.

$x = -5$; $x = 4$ makes f' undef.

$\uparrow$ not in $[-4, 7]$.

Let's make a list:

end points $\begin{cases} f(-4) = 1^{2/3} \cdot 8^{1/3} = 2 \\ f(7) = 12^{2/3} \cdot (-3)^{1/3} = -3 \cdot 2^{4/3} \end{cases}$

critical points in $[-4, 7]$ $\begin{cases} f(1) = 6^{2/3} \cdot 3^{1/3} = 3 \cdot 2^{2/3} \\ f(4) = 0 \end{cases}$

So, $f(x)$ has absolute max at $x = 1$ and $f(1) = 3 \cdot 2^{2/3}$

$f(x)$ has absolute min at $x = 7$ and $f(7) = -3 \cdot 2^{4/3}$

4. *(20 pts)* Use the guidelines to sketch the graph of $f(x) = \int_0^x -\frac{2t}{(t^2-1)^2}\,dt$

- $D_f : \mathbb{R} - \{-1, 1\}$ we can define the integral except $x=1$ and $x=-1$.

- $f(0) = 0$ is the y-intercept.

- Since $\frac{-t}{(t^2-1)^2}$ is an odd function, its integral is even.

  It should be symmetric with respect to y-axis.

- $$\lim_{x\to 1^-} \int_0^x -\frac{2t}{(t^2-1)^2}\,dt = \lim_{x\to 1^-} \frac{1}{x^2-1} + 1 = -\infty \underset{\text{symmetry}}{=} \lim_{x\to -1^+} f(x)$$

  $$\lim_{x\to 1^+} \int_0^x \frac{-2t\,dt}{(t^2-1)^2} = \lim_{x\to 1^+} \frac{1}{x^2-1} + 1 = +\infty \underset{\text{symmetry}}{=} \lim_{x\to -1^-} f(x)$$

  vertical asym.

  $$\lim_{x\to +\infty} \int_0^x \frac{-2t\,dt}{(t^2-1)^2} = \lim_{x\to\infty} \frac{1}{x^2-1} + 1 = 1 = \lim_{x\to -\infty} f(x)$$

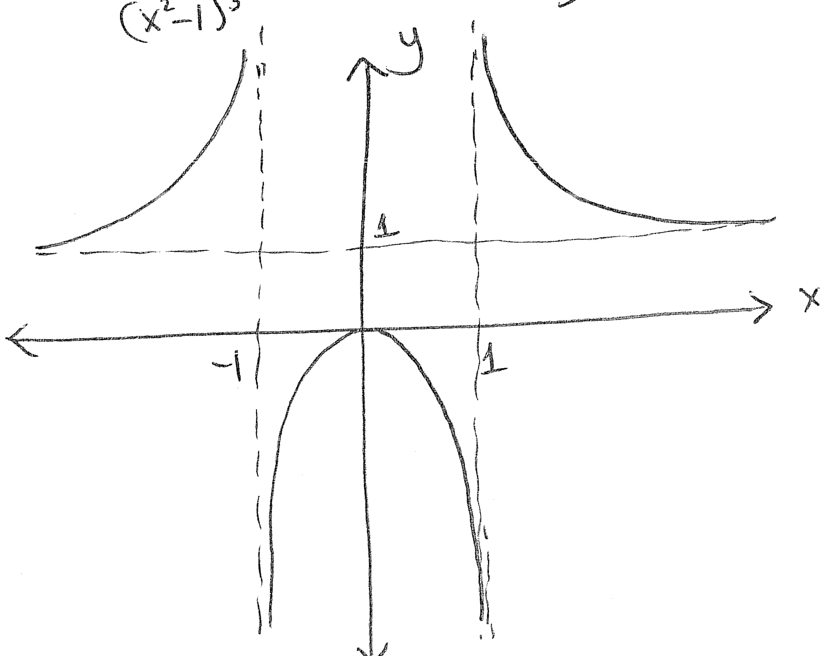
  Horizontal asym.

- $$f'(x) = -\frac{2x}{(x^2-1)^2}$$

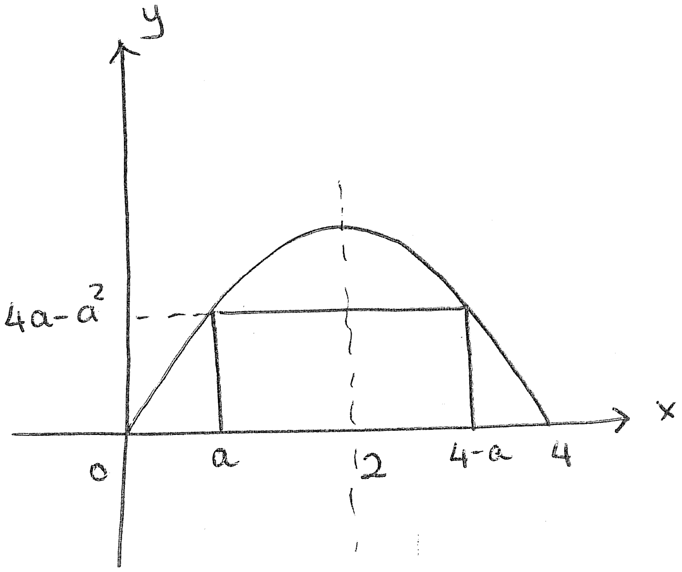
- $$f''(x) = -\left[\frac{2(x^2-1)^{\cancel{2}} - 2\cancel{(x^2-1)}\cdot(2x)^2}{(x^2-1)^{\cancel{4}\,3}}\right]$$

  $$= \frac{6x^2+2}{(x^2-1)^3}$$

x		-1		0		1	
f'	$+$		$+$		$-$		$-$
f''	$+$		$-$		$-$		$+$



5. *(15 pts)* Two vertices of a rectangle on the curve $y = 4x - x^2$ and other two vertices on the x-axis. Find the sides of the rectangle which has the largest circumference.



Circumference: $2(4-a-a) + 2(4a-a^2)$

$$C(a) = 8 - 4a + 8a - 2a^2$$

$$C(a) = 8 + 4a - 2a^2$$

$$C'(a) = 4 - 4a = 0 \Rightarrow a = 1.$$

(polynomial, can not be undefined!)

End points are: $a = 0$ and $a = 2$

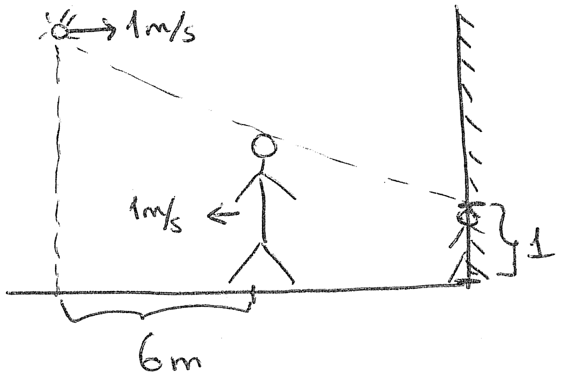
end points $\begin{cases} C(0) = 8 \\ C(2) = 8 \end{cases}$

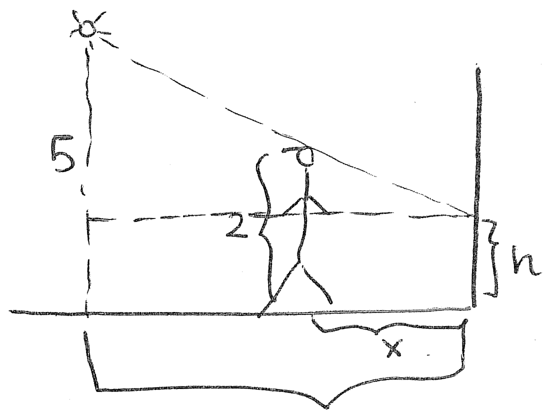
$C(1) = 10 \Rightarrow$ So the sides are: $4 - 2 \cdot 1 = 2$ and

$4 \cdot 1 - 1^2 = 3$

6. *(15 pts)* While a 5m tall light source is approaching to a wall by 1m/sec, 2m tall man between the source and a wall is also approaching the source by 1m/sec. When the distance between the light source and the man is 6m, the height of man's shadow is 1m. How fast is the height of shadow changing at that moment?

when $t=t_s$





Using similar triangles: $\dfrac{2-h}{5-h} = \dfrac{x}{y}$

$\Rightarrow \dfrac{-h'\cdot(5-h) - -h'(2-h)}{(5-h)^2} = \dfrac{x'\cdot y - y'x}{y^2}$

when $t=t_s$; $h=1$

and $y = x+6$

so, $\dfrac{2-1}{5-1} = \dfrac{x}{x+6}$

$\Rightarrow x = 2$ and $y=8$

when $t=t_s$

$\Rightarrow \dfrac{-h'\cdot(5-1)+h'(2-1)}{(5-1)^2} = \dfrac{1\cdot 8 - (-1)\cdot 2}{8^2}$

when $t=t$
$x' = 1$
$y' = -1$

$\dfrac{-3h'}{16} = \dfrac{10}{64} \Rightarrow h' = -\dfrac{5}{6}$ m/sec.

Bonus. Find the function $f(x)$ and value(s) of a satisfying the following equation,

$$\int_a^x \frac{f(t)}{t^3}\,dt = x^2 - a.$$

when $x=a$; $\displaystyle\int_a^a \frac{f(t)}{t^3}\,dt = a^2 - a \Rightarrow a^2 - a = 0$

$a = 0; a = 1.$

using F.T.C. $\left(\displaystyle\int_a^x \frac{f(t)}{t^3}\,dt\right)' = (x^2-a)'$

$\dfrac{f(x)}{x^3} = 2x \Rightarrow f(x) = 2x^4.$