6. *(15pts)* An ice cube is melting down so that its surface area decreases by $5cm^2$ per second. Find the rate at which the volume of the ice cube is decreasing when the side length of the cube is 3cm.

$a(t)$: each side of the cube at any time.

Area of the cube $A(t) = 6a^2(t) \Rightarrow A'(t) = 12a(t)a'(t)$

Volume of the cube $V(t) = a^3(t) \Rightarrow V'(t) = 3a^2(t)a'(t)$

The moment, when each side is 3 say t_m.

At that moment ; $A'(t_m) = 12a(t_m)\,a'(t_m)$

$$-5 = 12 \cdot 3 \cdot a'(t_m)$$

$$\Rightarrow a'(t_m) = -\frac{5}{36} \text{ cm/sec.}$$

So, $V'(t_m) = 3\,a^2(t_m) \cdot a'(t_m)$

$$= 3 \cdot 3^2 \cdot -\frac{5}{36} = -\frac{15}{4} \text{ cm}^3\text{/sec.}$$

Volume is decreasing by $\frac{15}{4}$ cm^3 per second.

METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY
MIDTERM 2

Code : *MAT 119*	Last Name:
Acad.Year: *2012-2013*	Name : KEY
Semester : *Spring*	Department:
Date : *23.03.2013*	Signature:
Time : *14:40*	Student No.:
Duration : *90 minutes*	Section:
	8 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1. (15)	2. (6)	3. (6)	4. (13)	5. (15)	6. (15)	7. (20)	8. (10)	Bonus	

Show your work! Please draw a $\boxed{\text{box}}$ around your answers!

1. *(3×5pts)* Compute the following limits. **DO NOT USE L'HOSPITAL!**

(a) $\lim_{x\to1} \frac{(x^2-1)(x+1)}{x^3-1} = \lim_{x\to1} \frac{\cancel{(x-1)}(x+1)(x+1)}{\cancel{(x-1)}(x^2+x+1)} = \frac{2 \cdot 2}{1+1+1} = \frac{4}{3}$

(b) $\lim_{x\to0} \frac{x-|x|}{2x+|x|}$ Check: $\lim_{x\to0^-} \frac{x-|x|}{2x+|x|} = \lim_{x\to0^-} \frac{x-(-x)}{2x+(-x)} = \lim_{x\to0^-} \frac{2x}{x} = 2$

$\lim_{x\to0^+} \frac{x-|x|}{2x+|x|} = \lim_{x\to0^+} \frac{x-(x)}{2x+(x)} = \lim_{x\to0^+} \frac{0}{3x} = 0$

Since left hand limit and right hand limit are different, limit D.N.E.

(c) $\lim_{x\to0} \cos\left(\frac{\sin x}{x}\right) = \cos\left(\underbrace{\lim_{x\to0} \frac{\sin x}{x}}_{1}\right) = \cos 1$

Since $\cos(x)$ is continuous

2. *(6pts)* Find the derivative of $f(x) = [g(x^2)]^3$ in terms of $g(x)$ and $g'(x)$.

$f'(x) = 3(g(x^2))^2 \cdot g'(x^2) \cdot 2x$

Chain Rule

3. *(6pts)* Find the derivative of $f(x) = \frac{(x+1)^2 \sin(x^3)}{\sqrt[5]{x}-1}$.

$$f'(x) = \frac{[(x+1)^2 \cdot \sin(x^3)]' \cdot (\sqrt[5]{x}-1) - \frac{1}{5}x^{-\frac{4}{5}} \cdot (x+1)^2 \cdot \sin(x^3)}{(\sqrt[5]{x}-1)^2}$$

Quotient Rule

$$= \frac{[2(x+1)\sin(x^3) + (x+1)^2 \cdot \cos(x^3) \cdot 3x^2](\sqrt[5]{x}-1) - \frac{x^{-\frac{4}{5}}}{5}(x+1)^2 \sin(x^3)}{(\sqrt[5]{x}-1)^2}$$

Product Rule

Chain Rule is also used in many terms.

4. *(13pts)* Find the equation of the tangent line to the curve $y^3x^2 + y^2x - x = 4$ at the point $(2,1)$.

Tangent line eqn must be in this form: $y = mx + b$

where m is just $y'|_{(2,1)}$, so by taking the derivative of given eqn;

$$3y^2 \cdot y' \cdot x^2 + y^3 2x + 2yy'x + y^2 - 1 = 0 \Rightarrow y' = \frac{1 - y^2 - 2y^3x}{3y^2x^2 + 2xy}$$

$$m = y'|_{(2,1)} = \frac{1 - 1^2 - 2 \cdot 1^3 \cdot 2}{3 \cdot 1^2 \cdot 2^2 + 2 \cdot 2 \cdot 1} = -\frac{1}{4}$$

Now, line eqn becomes; $y = -\frac{1}{4}x + b$.

To find b, we use common point information, which is $(2,1)$ is not only on the curve but also on the line.

Therefore, $1 = -\frac{1}{4} \cdot 2 + b \Rightarrow b = \frac{3}{2}$

Tangent line eqn: $y = -\frac{1}{4}x + \frac{3}{2}$

5. *(15pts)* Suppose that $f(x)$ is a continuous function such that $f(-1) = -1$, $f(0) = 1$ and $f(2) = -2$. Moreover $f'(0) = 0$ and $f'(x) \neq 0$ when $x \neq 0$. Show that the equation $f(x) = 0$ has **exactly** two solutions.

$\left.\begin{array}{l} f(-1) = -1 < 0 \\ f(0) = 1 > 0 \end{array}\right\}$ Since f is continuous, using I.V.T, f should have at least one root in $(-1,0)$

$\left.\begin{array}{l} f(0) = 1 > 0 \\ f(2) = -2 < 0 \end{array}\right\}$ Again, using I.V.T, f should have at least one root in $(0,2)$.

$(-1,0)$ and $(0,2)$ are two disjoint intervals having at least one root in each of them, hence f should have at least two roots in $(0,2)$.

Now, let's assume f has more than two roots; so we can pick at least three of them say: x_1, x_2, x_3.

So, $f(x_1) = f(x_2) = f(x_3) = 0$

Since f is differentiable we can use M.V.T on $[x_1, x_2]$;

there exists c_{12} in (x_1, x_2) such that $f'(c_{12}) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$.

Again, we can also use M.V.T on $[x_2, x_3]$;

there exists c_{23} in (x_2, x_3) such that $f'(c_{23}) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = 0$

This means f' should have two roots. However it is impossible by the definition of f whose derivative is 0 only at $x = 0$.

Therefore, f can not have more than two roots.

In conclusion, f has exactly two roots

7. *(20pts)* Find the maximum and minimum values of $f(x) = (x+2)^{1/3}(x-7)^{2/3}$ on the interval $[-1, 8]$.

To find critical points we should check where $f'(x)=0$ or undefined.

$$f'(x) = \frac{1}{3}(x+2)^{-\frac{2}{3}} \cdot (x-7)^{\frac{1}{3}} + \frac{2}{3}(x-7)^{-\frac{1}{3}} \cdot (x+2)^{\frac{1}{3}} = \frac{x-1}{(x+2)^{\frac{2}{3}}\,(x-7)^{\frac{1}{3}}}$$

Product Rule

After equating the denominators.

So, the critical points are : $x = 1$ (making $f'(x)=0$)

$x=-2$, $x=7$ (making $f'(x)$ undefined)

But, only $x=1$ and $x=7$ in our interval $[-1,8]$

Our list :

critical points:	end points:
$f(1) = 3^{\frac{1}{3}} \cdot (-6)^{\frac{2}{3}} = \sqrt[3]{108}$	$f(-1) = 1^{\frac{1}{3}} \cdot (-8)^{\frac{2}{3}} = \sqrt[3]{64}$
$f(7) = 0$	$f(8) = 10^{\frac{1}{3}} \cdot 1^{\frac{2}{3}} = \sqrt[3]{10}$

So, f has absolute max at $x=1$, and its value : $\sqrt[3]{108}$

f has absolute min at $x=7$, and its value : 0.

8. *(10pts)* Let

$$f(x) = \begin{cases} -x^2 + 2x + 1 & \text{when } x < 1 \\ x^2 + bx + c & \text{when } x \geq 1 \end{cases}$$

Find the values of b and c that make $f(x)$ continuous and differentiable at $x = 1$.

$f(x)$ is continuous: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$$\lim_{x \to 1^-} -x^2 + 2x + 1 = \lim_{x \to 1^+} x^2 + bx + c \Rightarrow 2 = 1 + b + c$$

$$\Rightarrow b + c = 1$$

$f(x)$ is differentiable: $\lim_{x \to 1^-} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1}$

$$\lim_{x \to 1^-} \frac{(-x^2 + 2x + 1) - (1 + b + c)}{x-1} = \lim_{x \to 1^+} \frac{(x^2 + bx + c) - (1 + b + c)}{x-1}$$

$$\lim_{x \to 1^-} \frac{-x^2 + 2x + 1 - 2}{x-1} = \lim_{x \to 1^+} \frac{x^2 + bx - 1 - b}{x-1}$$

$$\lim_{x \to 1^-} \frac{-(x-1)^{\cancel{2}}}{\cancel{x-1}} = \lim_{x \to 1^+} \frac{\cancel{(x-1)}(x+1+b)}{\cancel{x-1}}$$

$$0 = 2 + b \Rightarrow b = -2$$

$$\Rightarrow c = 3$$

Bonus. Write a function which is continuous everywhere but not differentiable everywhere.

$$f(x) = |x| = \begin{cases} -x & \text{when } x < 0 \\ x & \text{when } x \geqslant 0 \end{cases}$$

f is continuous everywhere: $f(x) = -x$ or $f(x) = x$ are polynomials and continuous everywhere we need to check $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$ ✓

$$\lim_{x \to 0^-} -x = \lim_{x \to 0^+} x = 0$$

f is not differentiable at $x = 0$: $\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$

$$\lim_{x \to 0^-} \frac{-x - 0}{x - 0} = \lim_{x \to 0^+} \frac{x - 0}{x - 0}$$