METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY
MIDTERM 2

Code : *MAT 119*	Last Name:
Acad. Year: *2012-2013*	Name :
Semester : *FALL*	Student No.:
Date : *22.12.2012*	Department:
Time : *9:40*	Section:
Duration : *110 minutes*	Signature:

6 QUESTIONS ON 6 PAGES
TOTAL 100 POINTS

1. (12)	2. (20)	3. (20)	4. (21)	5. (12)	6. (15)	Bonus		

Show your work! Please draw a $\boxed{\text{box}}$ around your answers!

1. *(4×3pts)* Compute the following limits.

(a) $\lim_{x\to\infty} \frac{(\ln x)^2}{x} \underset{=}{\text{L'HR}} \lim_{x\to\infty} \frac{2.\ln x \frac{1}{x}}{1} \underset{=}{\text{L'H.R}} \lim_{x\to\infty} \frac{2.\frac{1}{x}}{1} = 0$

(b) $\lim_{x\to 1} \frac{\int_0^{\ln x} t\, e^t\, dt}{x^2-1} \underset{\text{FTC}}{\overset{\text{L'HR}}{=}} \lim_{x\to 1} \frac{\ln x\, \cancel{e^{\ln x}} \cdot \cancel{\frac{1}{x}}}{2x} = 0$

(c) $\lim_{x\to 0} (\arctan(x))^x = e^{\lim_{x\to 0} \ln(\arctan x)^x} = e^{\lim_{x\to 0} \frac{\ln(\arctan x)}{\frac{1}{x}}} \underset{=}{\text{L'HR}} e^{\lim_{x\to 0} \frac{\frac{1}{\arctan x} \cdot \frac{1}{1+x^2}}{-\frac{1}{x^2}}}$

$= e^{\lim_{x\to 0} \frac{-x^2}{(\arctan x)(1+x^2)}} \underset{=}{\text{LH'R}} e^{\lim_{x\to 0} \frac{-2x}{\cancel{\frac{1}{1+x^2}} \cdot \cancel{1+x^2} + \arctan x . 2x}}$

$= e^0 = 1.$

(d) $\lim_{x\to\infty} \arctan(x) \arcsin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \arctan(x) \cdot \lim_{x\to\infty} \arcsin\left(\frac{1}{x}\right)$

$= \frac{\pi}{2} \cdot 0$

$= 0$

2. *(5×4pts)* Evaluate the following integrals.

(a) $\displaystyle\int \frac{(1-x)^2}{x}\,dx = \int \frac{1-2x+x^2}{x}\,dx = \int \frac{1}{x} - 2 + x\,dx$

$= \ln|x| - 2x + \frac{x^2}{2} + C$

(b) $\displaystyle\int \cos x \sec^2(\sin x)\,dx = \int \sec^2 u\,du = \tan u + C$

say $\sin x = u$

$\cos x\,dx = du$

$= \tan(\sin(x)) + C$

(c) $\displaystyle\int \frac{x^5}{\sqrt{x^3+1}}\,dx = \int \frac{(u-1)}{\sqrt{u}}\,\frac{du}{3} = \frac{1}{3}\int \sqrt{u} - \frac{1}{\sqrt{u}}\,du = \frac{1}{3}\left(\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right) + C$

say $x^3+1 = u$

$3x^2\,dx = du$

$= \frac{2}{9}(x^3+1)^{3/2} - \frac{2}{3}(x^3+1)^{1/2} + C$

(d) $\displaystyle\int_0^1 \frac{x\arctan(x^2)}{1+x^4}\,dx = \int_0^{\pi/4} u\,\frac{du}{2} = \left.\frac{u^2}{4}\right|_0^{\pi/4} = \frac{\pi^2}{64}$

say $\arctan(x^2) = u$

$\frac{1}{1+x^4}\cdot 2x\,dx = du$

when $x=0 \Rightarrow u = 0$

$x = 1 \Rightarrow u = \pi/4$

(e) $\displaystyle\int_{-1}^1 t^8\sin(t^5) + |t^2+2t|\,dt = \underbrace{\int_{-1}^1 t^8 \sin(t^5)\,dt}_{0} + \int_{-1}^1 |t^2+2t|\,dt$

Since $(-t)^8 \cdot \sin((-t)^5) = -t^8 \sin(t^5)$ it is odd function and the first integral becomes 0.

$= \displaystyle\int_{-1}^0 -t^2 - 2t\,dt + \int_0^1 t^2+2t\,dt$

$= \left.-\frac{t^3}{3} - t^2\right|_{-1}^0 + \left.\frac{t^3}{3} + t^2\right|_0^1 = \left(0 - \left(-\frac{2}{3}\right)\right) + \left(\frac{4}{3} + 0\right) = \frac{6}{3}$

3. *(5×4pts)* Find the following derivatives.

(a) $\frac{d}{dx}\left[2^{(x\ln x)}\right] = \ln 2 \cdot 2^{x\ln x} \cdot \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right)$

(b) $\frac{d}{dx}\left[\arcsin(\ln(1-x^2))\right] = \frac{1}{\sqrt{1-(\ln(1-x^2))^2}} \cdot \frac{1}{1-x^2} \cdot -2x$

(c) $\frac{d}{dx}\left[(\tan x)^{\arctan x}\right] = \frac{d}{dx}\left(e^{\ln(\tan x)^{\arctan x}}\right) = \frac{d}{dx} e^{\arctan x \cdot \ln(\tan x)}$

$= e^{\arctan x \cdot \ln(\tan x)} \cdot \left(\frac{1}{1+x^2} \cdot \ln(\tan x) + \arctan x \cdot \frac{1}{\tan x} \cdot \sec^2 x\right)$

(d) $(f^{-1})'(0)$ where $f(x) = \int_1^x \sqrt{3+t^3}\,dt$ $\Rightarrow f'(x) \overset{F.T.C}{=} \sqrt{3+x^3}$ and $f^{-1}(0) = 1$ (when $x=1$ integral is 0)

remember $(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))} \Rightarrow (f^{-1})'(0) = \frac{1}{\sqrt{3+(f^{-1}(0))^3}}$

$= \frac{1}{\sqrt{3+1}} = \frac{1}{2}$

(e) $\frac{d}{dx}\left[\int_{\cos(2)}^{\sin(x)} \ln(\sqrt{t^2+t})\,dt\right] \overset{F.T.C}{=} \ln\left(\sqrt{\sin^2(x)+\sin x}\right) \cdot \cos x$

4. *(3× 7pts)* In the following parts, R is the region between $y = x$ and $y = 2x - 1$ from $x = 0$ to $x = 2$.

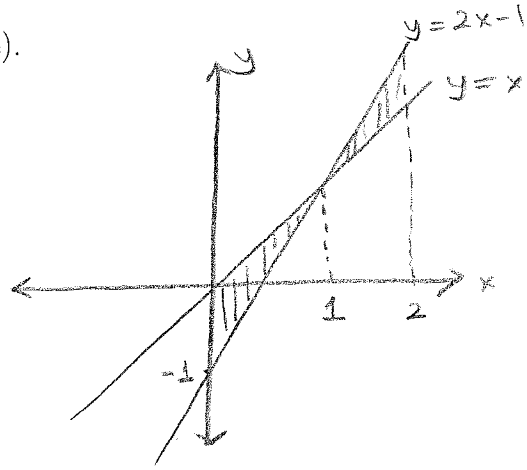
(a) Compute the area of the region R (described above).

$$\text{Area} = \int_0^2 |(2x-1) - x|\, dx$$

$$= \int_0^1 1 - x \, dx + \int_1^2 x - 1 \, dx$$

$$= x - \frac{x^2}{2}\Big|_0^1 + \frac{x^2}{2} - x\Big|_1^2$$

$$= \left(\frac{1}{2} - 0\right) + \left(0 - \left(-\frac{1}{2}\right)\right) = 1$$



(b) Compute the volume of R rotated around $y = -1$.

Using Disc Method;

$$\text{Volume} = \int_0^1 \pi \left[(x-(-1))^2 - ((2x-1)-(-1))^2\right] dx + \int_1^2 \pi \left[((2x-1)-(-1))^2 - (x-(-1))^2\right] dx$$

$$= \pi \int_0^1 \left[(x+1)^2 - (2x)^2\right] dx + \pi \int_1^2 \left[(2x)^2 - (x+1)^2\right] dx$$

$$= \pi \int_0^1 (x^2 + 2x + 1 - 4x^2)\, dx + \pi \int_1^2 (4x^2 - x^2 - 2x - 1)\, dx$$

$$= \pi\left(-x^3 + x^2 + x\right)\Big|_0^1 + \pi\left(x^3 - x^2 - x\right)\Big|_1^2$$

$$= \pi(1 - 0) + \pi(2 - (-1)) = 4\pi$$

(c) Compute the volume of R rotated around $x = -1$.

Using Shell Method;

$$\text{Volume} = \int_0^1 2\pi\,(x-(-1))\left(x - (2x-1)\right) dx + \int_1^2 2\pi\,(x-(-1))\left((2x-1) - x\right) dx$$

$$= \int_0^1 2\pi\,(x+1)(1-x)\, dx + \int_1^2 2\pi\,(x+1)(x-1)\, dx$$

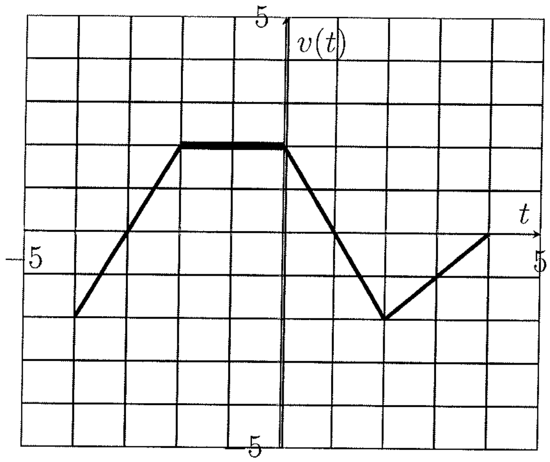
$$= 2\pi \int_0^1 (1 - x^2)\, dx + 2\pi \int_1^2 (x^2 - 1)\, dx$$

$$= 2\pi\left(x - \frac{x^3}{3}\right)\Big|_0^1 + 2\pi\left(\frac{x^3}{3} - x\right)\Big|_1^2$$

$$= 2\pi\left(\frac{2}{3} - 0\right) + 2\pi\left(\frac{2}{3} - \left(-\frac{2}{3}\right)\right)$$

$$= \frac{4\pi}{3} + \frac{8\pi}{3} = 4\pi$$

5. *(4×3pts)* The graph of a particle's vertical velocity function $v(t)$ is given below. Let $s(t)$ be the height of the particle at time t. Using this graph answer the following questions.



(a) Find the critical points of $s(t)$.

Since $s'(t) = v(t)$, critical points of $s(t)$ is where $v(t)$ is 0 or undefined. So, critical points are at $t=-3$, $t=1$ and $t=4$.

(b) Find local maximum and local minimum points of $s(t)$.

At the local max of $s(t)$; $s'(t)$ switches its sign from + to −.
At the local min of $s(t)$; $s'(t)$ switches its sign from − to +.
Since $s'(t) = v(t)$, local max of $s(t)$ at $t=1$, local min of $s(t)$ at $t=$

(c) Find the inflection points of $s(t)$.

At the inflection point of $s(t)$; $s''(t)$ switches from − to + or + to
Since $s'(t) = v(t)$ we should check $v'(t)$; so inflection point of $s(t)$ is at $t=2$.

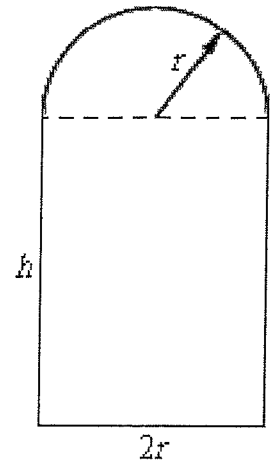
(d) If $s(-4) = 2$ then find $s(2)$.

Using F.T.C; $s(2) - s(-4) = \int_{-4}^{2} v(t)\,dt$

$$s(2) - 2 = \underbrace{-\cancel{\frac{2\cdot1}{2}} + \cancel{\frac{2\cdot1}{2}} + 2\cdot2 + \cancel{\frac{2\cdot1}{2}} - \cancel{\frac{2\cdot1}{2}}}_{\text{by calculating the areas}}$$

$\Rightarrow s(2) = 6$

6. *(15pts)* An arched window is being built. The bottom of the window is a rectangle and the top is a semicircle. What dimensions for the window will give the largest total area, if only 4 meters of framing material are available for the perimeter?



Length $= 2r+2h+\pi r = 4 \Rightarrow h = \frac{4-\pi r-2r}{2}$

Area $= \frac{\pi r^2}{2} + 2rh = \frac{\pi r^2}{2} + 2r\left(\frac{4-\pi r-2r}{2}\right)$

Area as a fonction of r: $A(r) = -2r^2+4r-\frac{\pi r^2}{2}$

That is defined in the interval $\left[0, \frac{4}{\pi+2}\right]$ ← when $h=0$.

To find critical point; $A'(r) = -4r+4-2\pi r = 0 \Rightarrow r = \frac{4}{4+\pi}$

To decide absolute max, we check: $A(0)=0$

$A\left(\frac{4}{4+\pi}\right) = \frac{8}{4+\pi} \approx 1.12$

$A\left(\frac{4}{\pi+2}\right) = \frac{8\pi}{(\pi+2)^2} \approx 0{,}95$

So, max area happens when;

$r = \frac{4}{4+\pi}$ and $h = \dfrac{4-\frac{4\pi}{4+\pi}-\frac{8}{4+\pi}}{2} = \frac{4}{4+\pi}$

Bonus. Find $f(x) \neq 0$ so that the region between $y=0$ and $y=f(x)$ from $x=1$ to $x=t$ has the same volume when rotated around $y=-1$ and when rotated around $x=-2$ (for all $t>0$).

Volume integrals must be the same;

$$\int_1^t \pi\left((f(x)+1)^2-1^2\right)dx = \int_1^t 2\pi(x-(-2))f(x)\,dt.$$

Using F.T.C;

$$\cancel{\pi}\left((f(t)+1)^2-1\right) = 2\cancel{\pi}(t+2)f(t)$$

$$f^2(t)+2f(t) = 2tf(t)+4f(t)$$

$$f(t)\left(f(t)-(2t+2)\right) = 0$$

So, $f(t)=0$ or $f(t)=2t+2$. Since $f(t)\neq 0$, then there is only one solution; $f(t)=2t+2$.