METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES MIDTERM 1	
Code : *MAT 120* Acad.Year: *2014-2015* Semester : *SPRING* Date : *28.03.2015* Time : *09:40* Duration : *120 min*	Last Name: Name : Student # : Signature : 6 QUESTIONS ON 5 PAGES TOTAL 100 POINTS

1. (20)	2. (12)	3. (12)	4. (21)	5. (15)	6. (20)		

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(7+8+5=20 pts)*

(A) Write down the equation of the line L that goes through the points $P(1,1,0)$ and $Q(0,3,1)$. Find the point S in which L intersects the xz-plane.

$\vec{QP} = \langle 1,-2,-1\rangle \Rightarrow L: \dfrac{x-1}{1} = \dfrac{y-1}{-2} = \dfrac{z-0}{-1}$

To find S take $y=0$ then, $\dfrac{x-1}{1} = \dfrac{0-1}{-2} = \dfrac{z}{-1}$

$\Rightarrow x = \frac{3}{2};\ z = -\frac{1}{2} \Rightarrow S$ is $\left(\frac{3}{2}, 0, -\frac{1}{2}\right)$

(B) Write down the equation of the plane that contains the point Q and line $y = x$ in the xy-plane.

Direction vector of the line is $\vec{u} = \langle 1,1,0\rangle$.
Since $O(0,0,0)$ is on the line, $\vec{OQ} = \langle 0,3,1\rangle$ then $\vec{u}\times\vec{OQ}$ must be perpendicular to the plane. $\vec{u}\times\vec{OQ} = \langle 1,-1,3\rangle$.

Plane eqn: $1(x-0) - 1(y-3) + 3(z-1) = 0$.

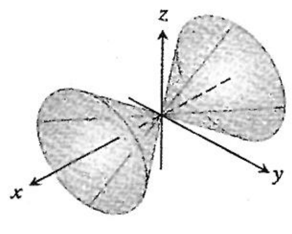
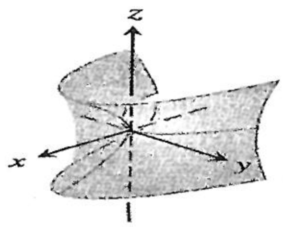
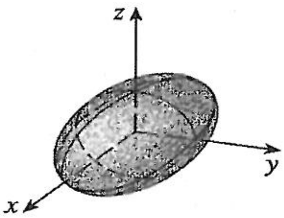
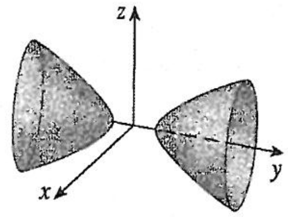
or $x - y + 3z = 0$

(C) Compute the distance between the point $R(0,0,1)$ and the plane in part B.

The distance between R and the plane is:

$$d_R = \frac{|0-0+3|}{\sqrt{1^2+(-1)^2+3^2}} = \frac{3}{\sqrt{11}}$$

2. *(12 pts)* For the given quadratic surfaces below, match the picture with a suitable equation from the list under the table and write the correct name of the surface like "elliptic cylinder".

	**Equation:** $2x^2 = y^2 + z^2$ **Name:** Circular Cone		**Equation:** $x = z^2 - 3y^2$ **Name:** Hyperbolic Paraboloid
	**Equation:** $2x^2 + 9y^2 + 4z^2 = 1$ **Name:** Ellipsoid		**Equation:** $\frac{x^2}{4} - y^2 + \frac{z^2}{6} = -1$ **Name:** Hyperboloid of 2 sheets

- $z = x^2 - 3y^2$
- $\frac{x^2}{4} - \frac{y^2}{9} + \frac{z^2}{3} = 5$
- $2x^2 = y^2 + z^2$
- $\frac{x^2}{4} - \frac{y^2}{9} + \frac{z^2}{3} = 5$
- $\frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{3} = 2$
- $z^2 = x^2 + y^2$
- $\frac{x^2}{4} + y^2 + \frac{z^2}{4} = 1$
- $x = z^2 - 3y^2$
- $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{12} = 5$
- $x = 4y^2 + z^2$
- $2x^2 + 9y^2 + 4z^2 = 1$
- $\frac{x^2}{4} - y^2 + \frac{z^2}{6} = -1$

3. *(8+4=12 pts)* This problem has two unrelated parts.

(A) Write the equation of the tangent line to the curve $\vec{\mathbf{r}}(t) = < 2\cos t, 2\sin t, e^t >$ $0 \leqslant t \leqslant \pi$ at the point where the tangent line is parallel to the plane $\sqrt{3}x + y = 1$.

Direction of the tangent line is : $\vec{r}'(t) = \langle -2\sin t, 2\cos t, e^t \rangle$. If it is parallel to the plane, then it must be perpendicular to its normal vector which is $\vec{n} = \langle \sqrt{3}, 1, 0 \rangle$. So, their dot product must be 0.

$\Rightarrow -2\sqrt{3}\sin t + 2\cos t = 0 \Rightarrow \tan t = \frac{1}{\sqrt{3}} \Rightarrow t = \frac{\pi}{6}$.

$\vec{r}(\frac{\pi}{6}) = \langle \sqrt{3}, 1, e^{\pi/6} \rangle$ and $\vec{r}'(\frac{\pi}{6}) = \langle -1, \sqrt{3}, e^{\pi/6} \rangle$. Tangent line eqn: $\frac{x-\sqrt{3}}{-1} = \frac{y-1}{\sqrt{3}} = \frac{z - e^{\pi/6}}{e^{\pi/6}}$

(B) Write a parameterization of the curve of intersection of $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

$\left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} \Rightarrow \cos^2 t + z^2 = 1 \Rightarrow z^2 = 1 - \cos^2 t \Rightarrow z = \mp \sin t$

We have following two curves in the intersection

$C_1: \langle \cos t, \sin t, \sin t \rangle$

$C_2: \langle \cos t, \sin t, -\sin t \rangle$.

4. *(7+7+7=21 pts)* Find the following limits if they exist or explain why they do not exist.

(A) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + \arctan(y^2)}{x^2 - 3y^2}$

Direction 1: x - axis

$$\lim_{(x,0)\to(0,0)} \frac{x^2+0}{x^2-0} = 1$$

$\neq$

Direction 2: y-axis

$$\lim_{(0,y)\to(0,0)} \frac{0+\arctan(y^2)}{0-3y^2} \overset{LHR}{=} \lim_{y\to 0} \frac{\frac{2\cancel{y}}{1+y^4}}{-6\cancel{y}} = -\frac{1}{3}$$

Limit does NOT exist.

(B) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\tan(x^{[redacted]}y^{[redacted]})}{x^2+y^2}$

Direction 1: x-axis

$$\lim_{(x,0)\to(0,0)} \frac{0}{x^2+0} = 0$$

$\neq$

Direction 2: y=x line on x-y plane.

$$\lim_{(x,x)\to(0,0)} \frac{\tan(x^2)}{2x^2} \overset{L'HR}{=} \lim_{x\to 0} \frac{\sec^2(x^2).2x}{4x} = \frac{1}{2}$$

Limit does NOT exist

(C) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1-e^{(x^2+y^2)}}{\sin(x^2+y^2)}$

Let $u = x^2+y^2$ then

$$\lim_{(x,y)\to(0,0)} \frac{1-e^{(x^2+y^2)}}{\sin(x^2+y^2)} = \lim_{u\to 0} \frac{1-e^u}{\sin(u)}$$

$$\overset{L'HR}{=} \lim_{u\to 0} \frac{-e^u}{\cos(u)} = -1.$$

5. *(10+5=15 pts)* Consider the function

$$F(x,y) = \left(x^{1/4} + \frac{y^{1/3}}{2}\right)^{5/2}$$

(A) Write down a linear approximation for F valid for points close to $(81, 8)$.

$$L(x,y) = F(81,8) + \frac{\partial F}{\partial x}\Big|_{(81,8)} (x-81) + \frac{\partial F}{\partial y}\Big|_{(81,8)} (y-8)$$

Here, $F(81,8) = \left(81^{1/4} + \frac{8^{1/3}}{2}\right)^{5/2} = 32$

$$\frac{\partial F}{\partial x}\Big|_{(81,8)} = \frac{5}{2}\left(x^{1/4} + \frac{y^{1/3}}{2}\right)^{3/2} \cdot \frac{1}{4} \cdot x^{-3/4}\Big|_{(81,8)} = \frac{5}{27}$$

$$\frac{\partial F}{\partial y}\Big|_{(81,8)} = \frac{5}{2}\left(x^{1/4} + \frac{y^{1/3}}{2}\right)^{3/2} \cdot \frac{1}{6} y^{-2/3}\Big|_{(81,8)} = \frac{5}{6}$$

$$L(x,y) = 32 + \frac{5}{27}(x-81) + \frac{5}{6}(y-8)$$

(B) Give an approximate value for $F(81.3, 7.8)$ on the basis of this linear approximation.

Using the linear approximation above,

$$F(81.3, 7.8) \approx L(81.3, 7.8) = 32 + \frac{5}{27}(81.3-81) + \frac{5}{6}(7.8-8)$$

$$= 32 + \frac{1}{18} - \frac{1}{6}$$

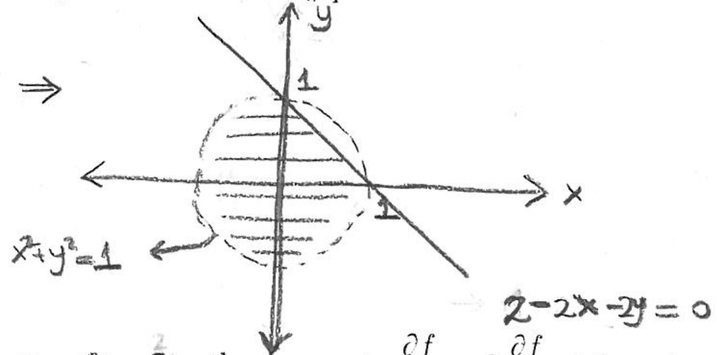
$$= 32 - \frac{1}{9}$$

6. *(6+8+6=20 pts)* This problem has two unrelated parts.

(A) Let $f(x, y) = \ln(1 - (x^2 + y^2)) - \sqrt{2 - 2x - 2y}$

(i) Find the domain of f and draw it on xy-plane.

$$\left.\begin{array}{l} 1-(x^2+y^2)>0 \\ 2-2x-2y \geq 0 \end{array}\right\} \Rightarrow$$



(ii) If $x = r^2 s + t$, $\quad y = e^r s - 2ts$, then compute $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial t}$ at $(r, s, t) = (0, 1, \frac{1}{2})$.

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} = \left[\left(\frac{-2x}{1-x^2-y^2}\right) + \frac{1}{\sqrt{2}\sqrt{1-x-y}}\right]\cdot 2rs + \left[\frac{-2y}{1-x^2-y^2} + \frac{1}{\sqrt{2}\sqrt{1-x-y}}\right]\cdot e^r s$$

at $(r,s,t) = (0,1,\frac{1}{2})$, $\quad x = \frac{1}{2}$, $y = 0$

$$\Rightarrow \frac{\partial f}{\partial r} = 1$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} = \left[\frac{-2x}{1-x^2-y^2} + \frac{1}{\sqrt{2}\sqrt{1-x-y}}\right]\cdot 1 + \left[\frac{-2y}{1-x^2-y^2} + \frac{1}{\sqrt{2}\sqrt{1-x-y}}\right]\cdot -2s$$

at $(r,s,t) = (0,1,\frac{1}{2})$, $\quad x = \frac{1}{2}$, $y = 0$

$$\Rightarrow \frac{\partial f}{\partial t} = -\frac{4}{3} + 1 - 2 = -\frac{7}{3}$$

(B) Compute the partial derivatives $\dfrac{\partial x}{\partial y}$ at $(0, 1, 1)$ for $z^2 = e^x yz - 3x^2 y \ln(z)$

Let $F(x,y,z) = z^2 - e^x yz + 3x^2 y \ln(z)$ then $\dfrac{\partial x}{\partial y} = -\dfrac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}}$

$$\Rightarrow \frac{\partial x}{\partial y} = -\frac{-e^x z + 3x^2\ln(z)}{-e^x yz + 6xy\ln(z)}$$

$$\Rightarrow \left.\frac{\partial x}{\partial y}\right|_{(0,1,1)} = \frac{-1}{1} = -1.$$