METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES MIDTERM									
Code : *MAT 120* Acad.Year: *2014-2015* Semester : *SUMMER* Date : *15.7.2013* Time : *17:40* Duration : *120 min*					Last Name: Name : KEY Student # : Signature : 5 QUESTIONS ON 4 PAGES TOTAL 100 POINTS				
1. (14)	2. (21)	3. (15)	4. (20)	5. (30)					

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(8+8=16 pts)* This problem has two related parts.

(A) Find the equation of the line L, passing through A(-1,2,0) and B(2,4,1).

$$\vec{v} = \vec{AB} = \langle 2-(-1), 4-2, 1-0 \rangle = \langle 3, 2, 1 \rangle$$

$$x = -1 + t3, \quad y = 2 + t \cdot 2, \quad z = t$$

(B) Find the equation of the plane P, which includes the line L (found in part a) and passing through (3,0,-1). Put $\vec{u} = \vec{AC} = \langle 3-(-1), 0-2, -1-0 \rangle = \langle 4, -2, -1 \rangle$

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & -1 \\ 3 & 2 & 1 \end{vmatrix} = -7\vec{j} + 14\vec{k}$$ is the normal vector of the plane P. Hence

$$-7(y-2) + 14(z-0) = 0 \quad \text{or}$$

$$2z = y - 2$$

2. *(8+8+8=24 pts)* Find the following limits if they exist or explain why they do not exist.

(A) $\lim_{(x,y)\to(0,0)} \dfrac{x^2 + \arcsin(y^2)}{x^2 - 3y^2} = f(x,y)$

$y=0 \Rightarrow f(x,0) = 1 \Rightarrow \lim_{x\to 0} f(x,0) = 1,$

$x=0 \Rightarrow f(0,y) = -\dfrac{\arcsin(y^2)}{3y^2} \Rightarrow \lim_{y\to 0} f(0,y) = -\dfrac{1}{3}$

So, the limit does not exist.

(B) $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^3}{\sin(x^8 + y^4)}$ If $x=0$ or $y=0$, then $f(x,y)=0$.

But for $y = x^2$ we have $f(x,x^2) = \dfrac{x^8}{\sin(2x^8)} \Rightarrow$

$\lim_{x\to 0} f(x,x^2) = \lim_{x\to 0} \dfrac{1}{2} \dfrac{2x^8}{\sin(2x^8)} = \dfrac{1}{2} \neq 0.$

So, the limit does not exist.

(C) $\lim_{(x,y)\to(0,0)} \dfrac{1 - e^{(x^2+y^2-1)}}{x^2 + y^2 - 1} = \dfrac{1-e^{-1}}{-1}$

3. *(15 pts)* Find the equation of the plane which is tangent to the surface given by the following equation $x^3y + y^2z + (x+y)z^3 = 1$ at the point Q(1,-1,2). Put $f(x,y,z) = x^3y + y^2z + (x+y)z^3$. Then $\nabla f = \langle 3x^2y + z^3, x^3 + 2yz + z^3, y^2 + 3(x+y)z^2 \rangle$ and $(\nabla f)(Q) = \langle 5, 5, 1 \rangle$ is the normal vector.

So, $5(x-1) + 5(y+1) + (z-2) = 0$ or

$$5x + 5y + z = 2$$

4. *(20 pts)* Find the directional derivative of the function $f(x,y,z) = \cos(xy+z)$ at $P(1/2, \pi/2, \pi/4)$ in the direction of the vector $\mathbf{v} = (-1, -1, -1)$. Find the partial derivatives $\partial f/\partial s$ and $\partial f/\partial t$ of the function $f(x,y,z)$ if $x = s+t, y = st, z = s/t$.

Note that $\vec{u} = \langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$ is the direction of $\vec{v}$,

$\nabla f = \langle -\sin(xy+z)\cdot y, -\sin(xy+z)\cdot x, -\sin(xy+z) \rangle$ and

$(\nabla f)(P) = \langle -\pi/2, -1/2, -1 \rangle$. It follows that

$$(D_{\vec{u}} f)(P) = (\nabla f)(P)\cdot \vec{u} = \frac{\pi}{2\sqrt{3}} + \frac{1}{2\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{\pi+3}{2\sqrt{3}}.$$

Finally, $\frac{\partial x}{\partial s} = 1, \frac{\partial y}{\partial s} = t, \frac{\partial z}{\partial s} = \frac{1}{t}$ and

$$\frac{\partial f}{\partial s} = -\sin((s+t)st + \tfrac{s}{t}) - \sin((s+t)st + \tfrac{s}{t})\cdot t - \sin((s+t)st + \tfrac{s}{t})\cdot \frac{1}{t}.$$

Similarly, $\frac{\partial x}{\partial t} = 1, \frac{\partial y}{\partial t} = s, \frac{\partial z}{\partial t} = -\frac{s}{t^2}$, and

$$\frac{\partial f}{\partial t} = -\sin((s+t)st + \tfrac{s}{t}) - \sin((s+t)st + \tfrac{s}{t})\cdot s + \sin((s+t)st + \tfrac{s}{t})\cdot \frac{s}{t^2}$$

5. *(25 pts)* Consider the function $f(x,y) = x^2 - 2x\cos(y)$ on the unbounded region $0 \leqslant y \leqslant \pi$.

(a) Find critical points of $f(x,y)$ and classify them.

$$\begin{cases} f_x = 2x - 2\cos(y) = 0 \\ f_y = 2x\sin(y) = 0 \end{cases} \rightarrow x = 0 \text{ or } \sin(y) = 0,\ 0 \le y \le \pi$$

$x = 0 \downarrow \cos(y) = 0$; $\quad \sin(y)=0 \downarrow y = 0, \pi,\ x = \cos(y)$

$$C.P. = \{ (0, \pi/2),\ (1, 0),\ (-1, \pi) \}$$

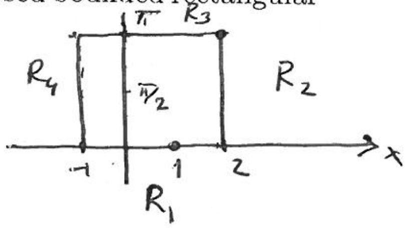
$$\Delta(x,y) = \begin{vmatrix} 2 & 2\sin(y) \\ 2\sin(y) & 2x\cos(y) \end{vmatrix} = 4x\cos(y) - 4\sin^2(y)$$

$\Delta(0, \pi/2) = -4 < 0 \Rightarrow (0, \pi/2)$ is a saddle point

$\Delta(1, 0) = 4 > 0 \Rightarrow (1, 0)$ is a local min. point.

$\Delta(-1, \pi) = 4 > 0 \Rightarrow (-1, \pi)$ is a local min. point.

(b) Find the absolute max and min values of the function $f(x,y)$ on the closed bounded rectangular region $R: -1 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant \pi$.



R_1: $y = 0 \Rightarrow f(x, 0) = x^2 - 2x$,

$f'(x, 0) = 2x - 2 \Rightarrow (-1, 0), (1, 0), (2, 0)$.

R_2: $x = 2,\ 0 \le y \le \pi \Rightarrow f(2, y) = 4 - 4\cos(y),\ f'(2, y) = 4\sin(y)$

$\Rightarrow (2, 0),\ (2, \pi)$.

R_3: $y = \pi,\ -1 \le x \le 2 \Rightarrow f(x, \pi) = x^2 + 2x,\ f'(x, \pi) = 2x + 2$

$\Rightarrow (2, \pi),\ (-1, \pi)$.

R_4: $x = -1,\ 0 \le y \le \pi \Rightarrow f(-1, y) = 1 + 2\cos(y),\ f'(-1, y) = -2\sin(y)$

$\Rightarrow (-1, 0),\ (-1, \pi)$.

$CP = \{ (-1, 0), (1, 0), (2, 0), (2, \pi), (-1, \pi) \}$

$f(-1, 0) = 3,\ f(1, 0) = -1,\ f(2, 0) = 0,\ f(2, \pi) = 8,\ f(-1, \pi) = -1$

$(-1, 0),\ (-1, \pi) \longrightarrow$ abs. min.

$(2, \pi) \longrightarrow$ abs. max