METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES FINAL EXAM	
Code : *MAT 120* Acad.Year: *2014-2015* Semester : *SUMMER* Date : *14.08.2015* Time : *9:00* Duration : *120 min*	Last Name: Name : Student # : Signature : 7 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1. (10)	2. (18)	3. (24)	4. (8)	5. (20)	6. (8)	7. (12)		

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(10pts)* Use Lagrange multipliers to find the maximum and minimum values of

$$f(x, y, z) = xyz \qquad \text{on} \qquad x^2 + 2y^2 + 3z^2 = 6.$$

Let

$f = xyz, \quad g = x^2 + 2y^2 + 3z^2 = 6.$

Look for (x,y,z) s.t. $g(x,y,z) = 6$ & $\nabla f = \lambda \nabla g$

(x). $f_x = \lambda g_x \Rightarrow yz = 2x \cdot \lambda$

(y) $f_y = \lambda g_y \Rightarrow xz = 4y \cdot \lambda$

(z) $f_z = \lambda g_z \Rightarrow xy = 6z \cdot \lambda$

NOTE: If x, y, z or λ is zero; then $f(p) = 0$. for some point p

Assume none of x, y, z or λ is zero.

Then; $\lambda = \dfrac{yz}{2x} = \dfrac{xz}{4y} = \dfrac{xy}{6z}$.

- $\dfrac{yz}{2x} = \dfrac{xz}{4y} \Rightarrow 2y^2 = x^2$

- $\dfrac{xz}{4y} = \dfrac{xy}{6z} \Rightarrow z^2 = \dfrac{2}{3}y^2$

- $x^2 + 2y^2 + 3z^2 = 6 \Rightarrow 2y^2 + 2y^2 + 2y^2 = 6$

  $y^2 = 1$

Therefore $y = \pm 1$

$x = \pm\sqrt{2}$ $\quad z = \pm\sqrt{\dfrac{2}{3}}$

$\text{MAX}(xyz) = \sqrt{2} \cdot 1 \cdot \dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{2}{\sqrt{3}}$

$\text{MIN}(xyz) = -\sqrt{2} \cdot 1 \cdot \dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{-2}{\sqrt{3}}$

NOTICE:

$-\dfrac{2}{\sqrt{3}} < 0 < \dfrac{2}{\sqrt{3}}$!

2. *(3×6=18pts)* Integrate the following double integral and line integrals.

(a) $\iint_D y\,dA$ where D is bounded by $y = x - 2$ and $x = y^2$.

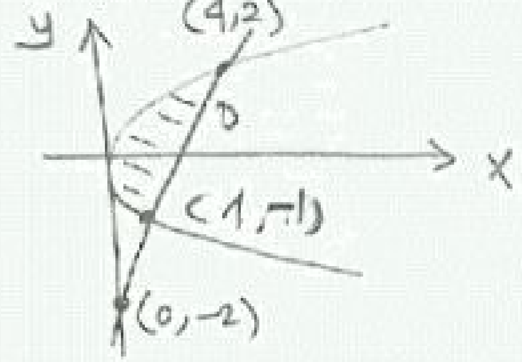
$X = y + 2 \qquad X = y^2$

Determine D:

$\left.\begin{array}{l} y = x-2 \\ x = y^2 \end{array}\right\} \quad y = y^2 - 2 \Rightarrow y^2 - y - 2 = 0$

$(y+1)(y-2) = 0$

$y = -1 \qquad y = 2.$



$$\iint_D y\,dA = \int_{y=-1}^{y=2}\int_{x=y^2}^{x=y+2} y\,dx\,dy$$

$$= \int_{-1}^{2} y\cdot(y+2-y^2)\,dy$$

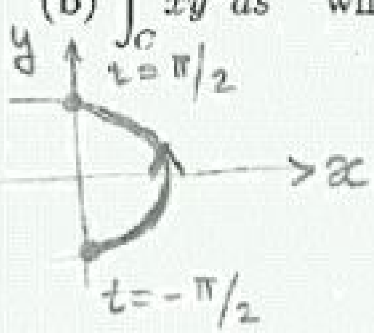
$$= \int_{-1}^{2} 2y + y^2 - y^3\,dy$$

$$= y^2 + y^3/3 - y^4/4 \Big|_{-1}^{2}$$

$$= (4-1) + \tfrac{1}{3}(8-(-1)) - \tfrac{1}{4}(16-1)$$

$$= 3 + 3 - 15/4 = \boxed{9/4}$$

(b) $\int_C xy^4\,ds$ where C is the right half of the circle $x^2 + y^2 = 16$.



$x = 4\cos t$

$y = 4\sin t$

$ds = \sqrt{x'(t)^2 + y'(t)^2}\,dt$

$ds = 4dt$

$$\int_C xy^4\,ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos t)(4\sin t)^4\,4dt$$

Change variables $\left\{\begin{array}{l} u = 4\sin t \\ du = 4\cos t\,dt \end{array}\right. \quad \begin{array}{l} t = \frac{\pi}{2} \Rightarrow u = 4 \\ t = -\frac{\pi}{2} \Rightarrow u = -4 \end{array}$

$$= 4\int_{-4}^{4} u^4\,du$$

$$= 4\cdot 2\cdot \int_0^4 u^4\,du$$

$$= 4\cdot 2\cdot \frac{u^5}{5}\Big]_0^4$$

$$\boxed{= 4\cdot 2\cdot \frac{4^5}{5}} = \frac{2}{5}\cdot 4^6$$

(c) $\int_C (x+y)^2 + 2x\,dy$ where C is the line segment from (2,1) to (3,0).

$$C = \left\{\begin{array}{l} x = 2+t \\ y = 1-t \end{array}\right. \qquad t \in [0,1] \qquad \begin{array}{l} dx = dt \\ dy = -dt \end{array}$$

$$\int_C (x+y)^2 + 2x\,dy = \int_0^1 3^2 + 2(2+t)\,(-dt)$$

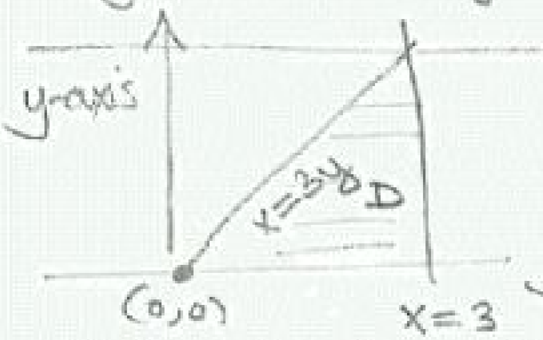
$$= -\int_0^1 9 + 4 + 2t\,dt = -\left[13t + t^2\right]_0^1 \boxed{= -14}$$

3. *(4×6=24pts)* Perform the indicated changes on the below integrals, BUT DO NOT INTEGRATE. (Problem continues on next page.)

(a) Reverse the order of integration. $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy \;=\; \iint_D e^{x^2}\, dA$

$y=1 \qquad x=3$

$y=0 \qquad x=3y \Rightarrow y=\frac{x}{3}$



$$\boxed{= \int_{x=0}^{x=3} \int_{y=0}^{y=\frac{x}{3}} e^{x^2}\, dy\, dx}$$

(b) Convert to polar coordinates. $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx \;=\; \iint_D \sqrt{x^2+y^2}\, dA$

Recall:

$x = r\cos\theta$

$y = r\sin\theta \qquad dA = r\,dr\,d\theta$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2\cos\theta} r \cdot r\,dr\,d\theta$$

What is D?

$x=2 \qquad y=\sqrt{2x-x^2}$

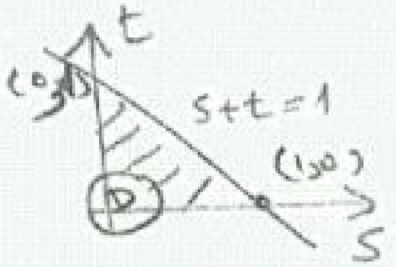
$x=0 \qquad y=0$

$$\boxed{\begin{aligned} &y=\sqrt{2x-x^2} \Rightarrow y^2 = 2x-x^2 \\ &x^2+y^2 = 2x \\ &r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta \end{aligned}}$$

(c) Change variables (x, y) to (s, t). $\iint_R (x-3y)\, dA$ where R is inside $y = 2x$, $2y = x$, and $y + x = 3$; with coordinates $x = 2s + t$ and $y = s + 2t$.

$y=2x$	$2y=x$	$y+x=3$
$s+2t = 2(2s+t)$	$2(s+2t) = 2s+t$	$s+2t+2s+t=3$
$3s=0$	$3t=0$	$3s+3t=3$
$\boxed{s=0}$	$\boxed{t=0}$	$\boxed{s+t=1}$

New Domain



$$\iint_R x-3y\, dA = \iint_D \{(2s+t) - 3(s+2t)\}\, 3\, dA$$

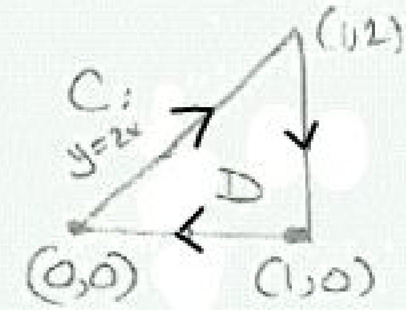
$$= 3\iint_D (-s-5t)\, dA$$

$$\frac{\partial(x,y)}{\partial(s,t)} = \det\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

$$= 3\int_0^1 \int_0^{1-s} -s-5t\, dt\, ds$$

(problem 3 continues here)

(d) Change to a double integral. $\int_C xy\,dx + x^2y^3\,dy$ where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$ oriented **clockwise**.



clockwise rotation

Apply Green's Theorem:

$$\int_C xy\,dx + x^2y^3\,dy = \iint_D \frac{\partial}{\partial x}(x^2y^3) - \frac{\partial}{\partial y}(xy)\,dA$$

$$= -\iint_D 2xy^3 - x\,dA = -\int_{x=0}^{x=1}\int_{y=0}^{y=2x} 2xy^3 - x\,dy\,dx$$

OR

$$= -\int_{y=0}^{y=2}\int_{x=\frac{y}{2}}^{x=1} 2xy^3 - x\,dx\,dy$$

4. *(2×4=8pts)* The following two parts are about the conservative vector field

$$\mathbf{F} = 2xe^{-y}\,\mathbf{i} + (2y - x^2e^{-y})\,\mathbf{j}$$

(a) Show that $\mathbf{F} = 2xe^{-y}\,\mathbf{i} + (2y - x^2e^{-y})\,\mathbf{j}$ is conservative.

$$\frac{\partial}{\partial y}(2xe^{-y}) = -2xe^{-y}$$

$$\|$$

$$\frac{\partial}{\partial x}(2y - x^2e^{-y}) = -2xe^{-y}$$

& F is defined on $\mathbb{R}^2$

Therefore, F is conservative on $\mathbb{R}^2$.

(b) Find the potential function $f(x,y)$ for $\mathbf{F} = 2xe^{-y}\,\mathbf{i} + (2y - x^2e^{-y})\,\mathbf{j}$. $= P\,i + Q\,j$

$$P = f_x = 2xe^{-y} \Rightarrow f = x^2e^{-y} + g(y)$$

$$\Downarrow$$

$$Q = f_y = 2y - x^2e^{-y} \qquad f_y = -x^2e^{-y} + g'(y)$$

$$g'(y) = 2y \Rightarrow g(y) = y^2 + c$$

$$\Rightarrow \boxed{f = x^2e^{-y} + y^2 + c}$$

5. *(5×4=20pts)* The following parts are about convergence/divergence tests for series.

(a) Write a series which is divergent by the test for divergence (n^{th} term test).

$$\sum_{n=1}^{\infty} \frac{n+1}{n+2}$$

Test for divergence:

$$\lim_{n\to\infty} \frac{n+1}{n+2} = 1 \neq 0$$

(b) Write a series which is convergent by the alternating series test.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$a_n = (-1)^n/n$

$b_n = \frac{1}{n}$

① $\lim_{n\to\infty} b_n = \lim \frac{1}{n} = 0$ ✓

② $n \le n+1 \Rightarrow \frac{1}{n+1} \le \frac{1}{n} \Rightarrow b_n$ is decreasing ✓

(c) Write a series which is not a p-series but is convergent by limit comparison with a p-series.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

clearly $\sum 1/n^2+1$ is not a p-series, but LCT with $\sum \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{1/(n^2+1)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 > 0$$

(d) Write a series which is is not geometric but is divergent by the ratio test.

$$\sum_{n=1}^{\infty} \frac{2^n}{n}$$

Ratio Test: $\lim_{n\to\infty} \frac{2^{n+1}/(n+1)}{2^n/n} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)$

$= 2 > 1$

(e) Is the series $\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$ convergent or divergent (state which tests are necessary).

$\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$ is absolutely convergent by comparison with $\sum_{n=1}^{\infty} \frac{1}{n!}$.

$\quad = (e-1)$

6. *(8pts)* Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{3^n (x+4)^n}{\sqrt{n}}$

Apply Ratio Test:

$$\lim_{n\to\infty} \left| \frac{3^{n+1} (x+4)^{n+1} / \sqrt{n+1}}{3^n (x+4)^n / \sqrt{n}} \right| = 3|x+4| < 1$$

$$|x+4| < \frac{1}{3}$$

$$\boxed{\text{Radius of convergence} = \frac{1}{3}.}$$

7. *(12pts)* Find the power series of $f(x) = x(1-x)^{-2}$.

$$f(x) = \frac{x}{(1-x)^2} \qquad \text{Recall } \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{(1-x)} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} n \cdot x^{n-1}$$

$$\frac{1}{1-x^2} = \sum_{n=1}^{\infty} n \cdot x^{n-1}$$

$$\boxed{\frac{x}{1-x^2} = \sum_{n=1}^{\infty} n \cdot x^n}$$