METU - NCC

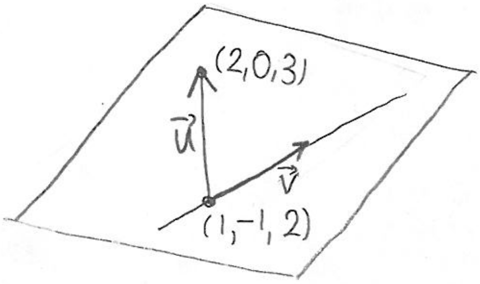
CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES MIDTERM	
Code : *MAT 120* Acad.Year: *2013-2014* Semester : *SUMMER* Date : *07.12.2013* Time : *17:00* Duration : *120 min*	Last Name: Name : Student # : KEY Signature : 7 QUESTIONS ON 5 PAGES TOTAL 100 POINTS

1. (14)	2. (14)	3. (10)	4. (12)	5. (10)	6. (20)	7. (20)	

Please draw a box around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(7+7=14 pts)* Let P be the plane passing through the point $(2,0,3)$, and containing the line $\dfrac{x-1}{2} = y+1 = \dfrac{z-2}{3}$.

(A) Find the equation of the plane P.



Direction vector of the line is $\vec{v} = \langle 2,1,3 \rangle$

$\vec{u} = (2,0,3) - (1,-1,2) = \langle 1,1,1 \rangle$

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -2i + j + k = \langle -2,1,1 \rangle$$

$$\langle -2,1,1 \rangle \cdot \langle x-2, y, z-3 \rangle = 0$$

$$-2x + y + z + 4 - 3 = 0 \Rightarrow -2x + y + z + 1 = 0$$

(B) Find the distance of the point $(3,0,1)$ to the plane P.

$$d = \frac{|-2\cdot 3 + 0 + 1 + 1|}{\sqrt{(-2)^2 + 1^2 + 1^2}} = \frac{4}{\sqrt{6}}$$

2. *(7+8=15 pts)* Find the given limits if they exist, or explain why they don't exist.

(A) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$

Along $x=0$ $\quad \displaystyle\lim_{y\to 0} \frac{\sin(0\cdot y)}{0^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$

$\not\approx$

Along $y=x$ $\quad \displaystyle\lim_{x\to 0} \frac{\sin(x\cdot x)}{x^2+x^2} = \lim_{x\to 0} \frac{\sin(x^2)}{2\cdot x^2} = \frac{1}{2}$

Limit doesn't exist.

(B) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2-x^3y^3}{x^2+y^2} = \lim_{(x,y)\to(0,0)} 1 - \frac{x^3y^3}{x^2+y^2} = 1-0 = 1$

$$0 \leqslant \left|\frac{x^3y^3}{x^2+y^2}\right| = \frac{x^2|x\cdot y^3|}{x^2+y^2} \leqslant |x\cdot y^3|$$

by squeeze thm.

since $x\cdot y^3$ is a polynomial

0

3. *(10 pts)* Find the equation of the tangent plane to the graph of $f(x,y) = x^2\sin(y) - x\cos^2(y)$ at the point $(1,0,-1)$.

$f_x = 2x\sin(y) - \cos^2(y) \qquad f_x(1,0) = -1$

$f_y = x^2\cos(y) + 2x\cos(y)\cdot\sin y \quad f_y(1,0) = 1$

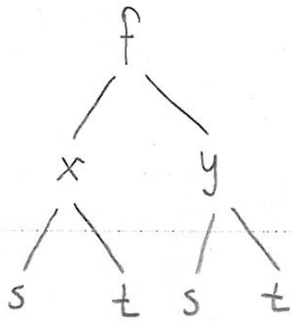
Hence, $\vec{n} = \langle -1, 1, -1\rangle$

$\langle -1,1,-1\rangle \cdot \langle x-1, y-0, z+1\rangle = 0$

$\boxed{-x+y-z=0}$

4. *(10 pts)* Let $z = f(x, y)$, and $x = t^2 + e^s$, $y = t\sin(s^2) + 1$ where $f(1,0) = 2$, $f_x(1,0) = -1$, $f_y(1,0) = 0$ and $f(2,1) = -2$, $f_x(2,1) = 1$, $f_y(2,1) = 2$

Compute the gradient of f at $(t, s) = (1, 0)$



$$\nabla f(1,0) = \langle f_t(1,0), f_s(1,0) \rangle$$

$$f_s = f_x \cdot x_s + f_y \cdot y_s = f_x(x,y) \cdot e^s + f_y(x,y) \cdot t \cdot \cos(s^2) \cdot 2s$$

$$f_s(1,0) = f_x(2,1)\, e^0 + f_y(2,1) \cdot 1 \cdot \cos(0) \cdot 2 \cdot 0 = 1$$

$$f_t = f_x \cdot x_t + f_y \cdot y_t = f_x(x,y) \cdot 2t + f_y(x,y) \cdot \sin(s^2)$$

$$f_t(1,0) = f_x(2,1) \cdot 2 \cdot 1 + f_y(2,1) \cdot \sin(0) = 2$$

$$\nabla f(1,0) = \langle 2, 1 \rangle$$

5. *(10 pts)* A climber is on a mountain with an equation $z = f(x, y)$ where z denotes the height. He is currently at the position $(a, b, f(a, b))$. If he walks in the direction of $\vec{u} = <\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}>$, then his height won't change and if he walks in the direction of $\vec{v} = <\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}>$, then he will start to descend at a rate of $3\sqrt{5}$. Find the maximum rate of descent and in which direction it occurs.

Let $\nabla f(a,b) = \langle k, l \rangle$

$$\left.\begin{aligned} D_{\vec{u}} f(a,b) &= \nabla f(a,b) \circ \vec{u} = \frac{k}{\sqrt{2}} - \frac{l}{\sqrt{2}} = 0 \Rightarrow k = l \\ D_{\vec{v}} f(a,b) &= \nabla f(a,b) \circ \vec{v} = \frac{k}{\sqrt{5}} - \frac{2l}{\sqrt{5}} = -3\sqrt{5} \Rightarrow k - 2l = -15 \end{aligned}\right\} \langle k, l \rangle = \langle +15, +15 \rangle$$

Maximum rate of descent occurs in the direction of $-\dfrac{\nabla f}{|\nabla f|} = \left\langle \dfrac{-15}{15\sqrt{2}}, \dfrac{-15}{15\sqrt{2}} \right\rangle = \left\langle -\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}} \right\rangle$ and the rate is equal to $-|\nabla f| = -15\sqrt{2}$.

6 (6+7+8=20 pts) This problem has three unrelated parts about the double integrals.

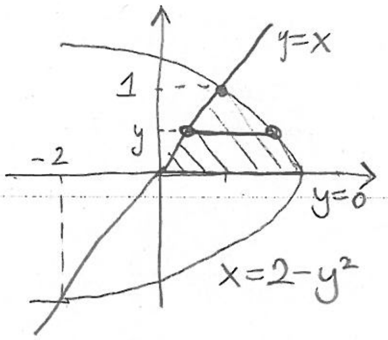
(A) Compute $\iint_R xy^2 + 2\, dA$ where R is the region bounded by $y = 0$, $y = x$, and $x = 2 - y^2$.

$$\int_0^1 \int_y^{2-y^2} xy^2+2\, dx\, dy = \int_0^1 \left(\left(\frac{x^2}{2} y^2 + 2x \right) \Big|_y^{2-y^2} \right) dy$$

$$= \int_0^1 \left(\frac{(2-y^2)^2}{2} y^2 + 2\cdot(2-y^2) - \frac{y^2 y^2}{2} - 2y \right) dy$$

$$= \int_0^1 \frac{y^6}{2} - \frac{5}{2} y^4 - 3y^2 - 2y + 4\, dy = \left(\frac{y^7}{14} - \frac{1}{2} y^5 - y^3 - y^2 + 4y \right) \Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{2} - 1 - 1 + 4 = \frac{7}{4}.$$



$2 - y^2 = y$
$0 = y^2 + y - 2$
$0 = (y+2)(y-1)$

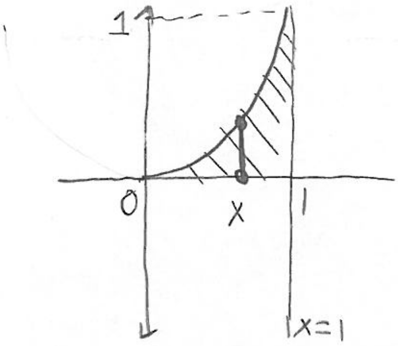
(B) Compute $\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\, dx\, dy$

$$\int_0^1 \int_0^{x^2} y \frac{e^{x^2}}{x^3}\, dy\, dx = \int_0^1 \left(\frac{y^2}{2} \frac{e^{x^2}}{x^3} \Big|_0^{x^2} \right) dx$$

$$= \int_0^1 \frac{x^4}{2} \cdot \frac{e^{x^2}}{x^3} - 0\, dx = \frac{1}{2} \int_0^1 x e^{x^2} dx = \frac{1}{4} \int_0^1 e^u du$$

$u = x^2$
$du = 2x dx$

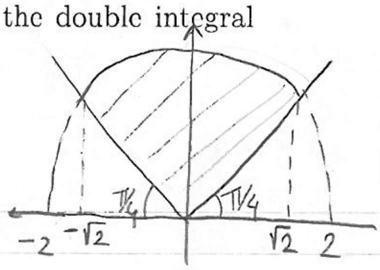
$$= \frac{1}{4} e^u \Big|_0^1 = \frac{1}{4}(e-1).$$



(C) Given $\iint_D \tan^{-1}(y/x) x^2\, dA$ where $D = \{(x, y) |\ x^2 + y^2 \leqslant 4,\ \ |x| \leqslant y\}$.

(i) Write an iterated integral in cartesian coordinates which is equal to the double integral above.

$$\int_{-\sqrt{2}}^{0} \int_{-x}^{\sqrt{4-x^2}} \tan^{-1}(y/x) \cdot x^2\, dy\, dx + \int_0^{\sqrt{2}} \int_x^{+\sqrt{4-x^2}} \tan^{-1}(y/x) x^2\, dy\, dx$$



(ii) Write an iterated integral in polar coordinates which is equal to the double integral above.

$$\int_{\pi/4}^{3\pi/4} \int_0^2 \theta \cdot r^2 \cdot \cos^2\theta \cdot r \cdot dr\, d\theta$$

7. *(7+8+5=20 pts)* Let $f(x,y) = x^2 + y^3 + y^2 - 1$

(A) Find the critical point(s) of $f(x,y)$ and classify as local maximum, local minimum or saddle.

$f_x = 2x = 0 \Rightarrow x = 0$

$f_y = 3y^2 + 2y = y(3y+2) = 0 \Rightarrow y = 0$ or $y = -2/3$ $\Big\}$ $(0,0)$ and $(0,-2/3)$ are critical points.

$f(0,0) = -1$

$f(0,-\frac{2}{3}) = -\frac{23}{27}$

$f_{xx} = 2$, $f_{xy} = f_{yx} = 0$, $f_{yy} = 6y + 2$

$\text{Hess}(f) = \begin{bmatrix} 2 & 0 \\ 0 & 6y+2 \end{bmatrix}$ $\quad \text{Hess}(f)(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $D = 4 > 0$, $2 > 0$ Local minimum

$\text{Hess}(f)(0,-2/3) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ $D = -4 < 0$ Saddle

(B) Use **Lagrange multipliers** to find the maximum and minimum values of $f(x,y)$ subject to $y + x^2 - 1 = 0$ where $-2 \leqslant x \leqslant 1$ $\quad 0 = g(x,y) = y + x^2 - 1$

$\nabla f = \langle 2x, 3y^2 + 2y \rangle \quad \nabla g = \langle 2x, 1 \rangle$

$\nabla f = \lambda \nabla g \Rightarrow$ (i) $2x = \lambda \cdot 2x \Rightarrow 2x(1-\lambda) = 0 \Rightarrow x = 0$ or $\lambda = 1$

(ii) $3y^2 + 2y = \lambda$

(iii) $y + x^2 - 1 = 0$

$x = 0$,

(iii) $\Rightarrow y = 1$, (ii) $\Rightarrow \lambda = 5$

$(0,1)$

$f(0,1) = 1$ ↳ Global Max.

$\lambda = 1$

(ii) $\Rightarrow 3y^2 + 2y - 1 = 0$

$(3y-1)\cdot(y+1) = 0$

$y = \frac{1}{3}$, $y = -1$

$x = \mp\sqrt{\frac{2}{3}}$ $\quad x = \mp\sqrt{2}$

$f(\mp\sqrt{\frac{2}{3}}, \frac{1}{3}) = -\frac{5}{27}$

$f(-\sqrt{2}, -1) = 1$

Boundaries

$x = -2$, $y = -3$

$f(-2,-3) = -15$

$x = 1$, $y = 0$

$f(1,0) = 0$

Global min.

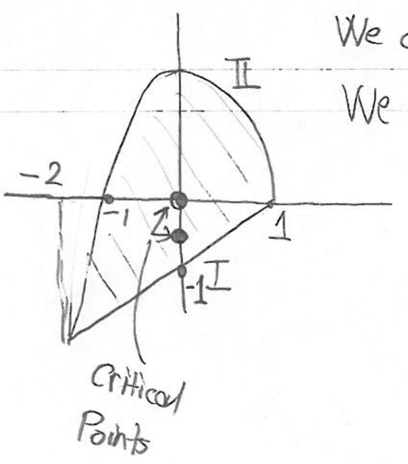
$f(x,y)$ is cont.

$y + x^2 - 1 \leq 0$

$-2 \leq x \leq 1$

is closed and bounded. By Ext. Val. Thm. Absolute max/min value exits.

(C) Find the maximum and minimum values of $f(x,y)$ over D $= \{(x,y) | y \leqslant 1 - x^2$ and $y \geqslant x - 1\}$



We checked critical points in (a), checked boundary II in (b)

We need to check I: $(y+1, y)$ $\quad -3 \leq y \leq 1$

$f(x,y) \underset{\text{on I}}{=} f(y) = y^2 + 2y + 1 + y^3 + y^2 - 1$

$= y^3 + 2y^2 + 2y$

$f'(y) = 3y^2 + 4y + 2 = (3y+2)(y+1) = 0$

$y = -2/3$, $y = -1$

$x = \mp\sqrt{\frac{5}{3}}$ $\quad x = \mp\sqrt{2}$

$f(-\sqrt{\frac{5}{3}}, -\frac{2}{3}) = \frac{22}{27}$

$f(-\sqrt{2}, -1) = 1$ → Global Max.

$f(-2,-3) = -15$ → Global min.