M E T U
Northern Cyprus Campus

	Calculus for Functions of Several Variables Midterm	

Code : *Math 120* Acad.Year: *2012-2013* Semester : *Summer* Date : *16.7.2013* Time : *17:40* Duration : *120 minutes*	Last Name: Name: Student No: Department: Section: Signature: 10 QUESTIONS ON 6 PAGES TOTAL 110 POINTS

1 (12)	2 (8)	3 (8)	4 (16)	5 (11)	6 (10)	7 (10)	8 (10)	9 (10)	10 (10)	B (5)		

Show your work! No calculators! Please draw a $\boxed{\text{box}}$ around your answers!

Please do not write on your desk!

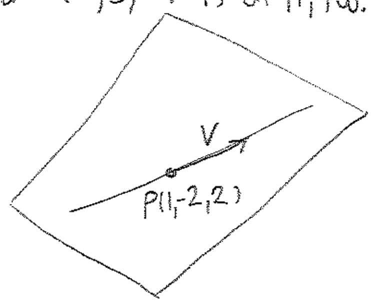
1 (This problem has two unrelated parts)

(a) (6 pts) Find an equation of the plane passing through the point (0,3,4) and containing the line $r(t) = < 1-t, t-2, 3t+2 >$.

$r(t) = \langle 1,-2,2\rangle + t\langle -1,1,3\rangle$, so $v = \langle -1,1,3\rangle$ is on the plane

$P(1,-2,2)$ and $Q(0,3,4)$ are two points on the plane. $\vec{PQ} = \vec{u} = \langle -1,5,2\rangle$ is on it, too.

$$\vec{n} = u \times v = \begin{vmatrix} i & j & k \\ -1 & 5 & 2 \\ -1 & 1 & 3 \end{vmatrix} = 13i + j + 4k$$



Plane Equation: $\langle 13,1,4\rangle \cdot \langle x, y-3, z-4\rangle = 0$

$$\boxed{13x + y + 4z - 19 = 0}$$

(b) (6 pts) Let L_1 be the line through the points $(1,0,0)$ and $(0,2,0)$, and L_2 be the line through the points $(0,-1,1)$ and $(0,0,3)$. Find the distance between L_1 and L_2.

$L_1: \; r_1(t) = \langle 1,0,0\rangle + t\langle -1,2,0\rangle = \langle 1-t, 2t, 0\rangle$

$L_2: \; r_2(s) = \langle 0,-1,1\rangle + s\langle 0,1,2\rangle = \langle 0, s-1, 2s+1\rangle$

They will lie on two parallel planes with normal $n = \langle -1,2,0\rangle \times \langle 0,1,2\rangle$

$$n = \begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 4i + 2j - k$$

So, L_1 lies on $\langle 4,2,-1\rangle \cdot \langle x-1, y, z\rangle = 0 \Rightarrow 4x + 2y - z - 4 = 0$

L_2 " " $\langle 4,2,-1\rangle \cdot \langle x, y+1, z-1\rangle = 0 \Rightarrow 4x + 2y - z + 3 = 0$

So, the distance between parallel planes $\dfrac{|-4-3|}{\sqrt{4^2+2^2+(-1)^2}} = \dfrac{7}{\sqrt{21}}$

which is the distance between these lines.

2.(4+4 pts) Let $\mathbf{r}_1(t) = < t, t^2, t^3 >$, $\mathbf{r}_2(s) = < 2s^3, 4s^2, 8s >$ be parametrizations of two curves C_1, C_2.

(a) Show that C_1 and C_2 intersect at exactly three points.

$$\left.\begin{array}{ll} \text{I}: & t = 2s^3 \\ \text{II}: & t^2 = 4s^2 \end{array}\right\} \quad 4s^6 = 4s^2 \Rightarrow 4s^2(s^4-1) = 0 \Rightarrow 4s^2(s^2-1)\cdot(s^2+1) = 0$$

$\text{III}: \ t^3 = 8s$

$s = 0, \ s = \mp 1$

$t = 0, \ t = \mp 2$

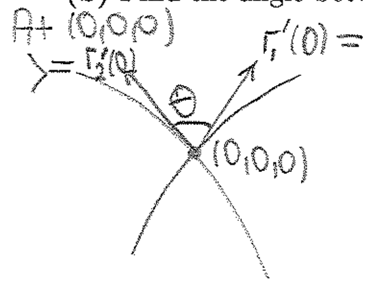
$\text{III}: \ 0^3 = 8 \cdot 0 \ ✓$

$2^3 = 8 \cdot 1 \ ✓$

$(-2)^3 = 8 \cdot (-1) \ ✓$

So, the points are $(0,0,0), (2,4,8), (-2,4,-8)$

(b) Find the angle between the two curves at one of these intersection points.



$r_1'(t) = \langle 1, 2t, 3t^2 \rangle \quad r_1'(0) = \langle 1, 0, 0 \rangle$

$r_2'(s) = \langle 6s^2, 8s, 8 \rangle \quad r_2'(0) = \langle 0, 0, 8 \rangle$

$$r_1'(0) \cdot r_2'(0) = |r_1'(0)| \cdot |r_2'(0)| \cdot \cos\theta$$

$$0 = 1 \cdot 8 \cdot \cos\theta \Rightarrow \theta = \frac{\pi}{2} \text{ or } 90^\circ$$

3.(4+4 pts) For the quadratic surface $x^2 - 2y^2 + z^2 = 0$.

(a) By describing its $x = 1, y = 1$ and $z = 0$ cross-sections, state the name of the surface.

$x = 1 \quad 1 = 2y^2 - z^2$ Hyperbola

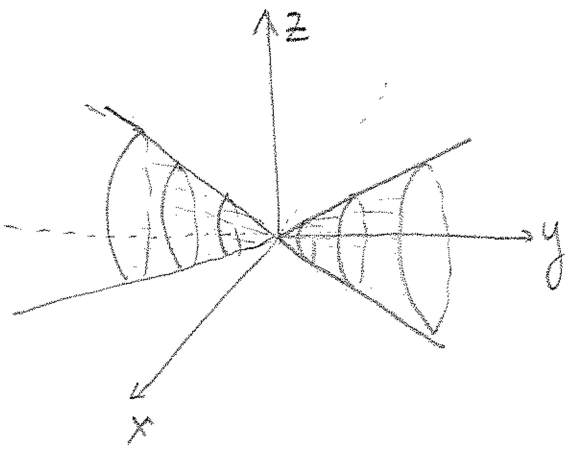
$y = 1 \quad x^2 + z^2 = 2$ Circle

$z = 0 \quad x^2 = 2y^2 \Rightarrow x = \mp\sqrt{2}y$ Pair of Lines

Circular Cone

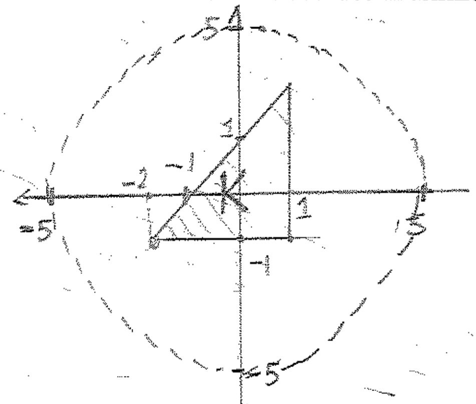
$2y^2 = x^2 + z^2$ in y-direction

(b) Graph the quadratic surface.



4. (4+6+6 pts) Let $f(x,y) = \dfrac{1}{x^2+y^2-25}$, and $K \subset \mathbb{R}^2$ the closed region bounded by the lines $y - x = 1$, $y = -1$ and $x = 1$.

(a) Sketch the region K and the domain D of $f(x,y)$. Show that $f(x,y)$ must have an absolute minimum and an absolute maximum on K.



$f(x,y)$ has domain $\mathbb{R}^2 \setminus \{x^2+y^2=5\}$

Since $f(x,y)$ is a rational function it's cont. in $Dom(f)$ which contains K

K is compact (closed and bounded).

Hence, $f(x,y)$ have absolute maximum and minimum on K.

(b) Find all critical points of $f(x,y)$ on the interior of K and classify them using the second derivative test.

$f_x = \dfrac{-2x}{(x^2+y^2-25)^2} = 0 \Rightarrow x = 0$ $\quad$ $(0,0)$ is a critical point.

"Local Maximum"

$f_y = \dfrac{-2y}{(x^2+y^2-25)^2} = 0 \Rightarrow y = 0$

$f_{xx} = \dfrac{-2(x^2+y^2-25)^2 + 2x(2(x^2+y^2-25)\cdot 2x)}{(x^2+y^2-25)^4}$

$f_{xy} = f_{yx} = \dfrac{+\,2x(2(x^2+y^2-25)\cdot 2y)}{(x^2+y^2-25)^2}$

$f_{yy} = \dfrac{-2(x^2+y^2-25)^2 + 2y(2(x^2+y^2-25)\cdot 2y)}{(x^2+y^2-25)^4}$

$$\text{Hesian}(f)(0,0) = \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} = \begin{bmatrix} \frac{-2}{25^2} & 0 \\ 0 & \frac{-2}{25^2} \end{bmatrix} \quad D = \frac{4}{25^4} > 0$$

$f_{xx} < 0$

(c) Find the absolute minima and maxima of $f(x,y)$ on K.

I: $y = -1$, $-2 \le x \le 1$ $\quad f(x) = \dfrac{1}{x^2-24}$

$f'(x) = \dfrac{-2x}{(x^2-24)^2} = 0 \quad x = 0$

$(0,-1)$ (Critical)

$(-2,-1)$, $(1,-1)$ (Boundary)

II: $y = x+1$, $-2 \le x \le 1$ $\quad f(x) = \dfrac{1}{2x^2+2x-24}$

$f'(x) = \dfrac{-(4x+2)}{(2x^2+2x-24)^2} = 0 \quad x = \dfrac{-1}{2}$

$\left(-\frac{1}{2}, \frac{1}{2}\right)$ (Critical)

$(-2,-1)$, $(1,2)$ (Boundary)

III: $x = 1$ $\quad -1 \le y \le 2$ $\quad f(y) = \dfrac{1}{y^2-24}$

$f'(y) = \dfrac{-2y}{(y^2-24)^2} = 0 \quad y = 0$

$(1,0)$ (Critical

$(1,2)$, $(1,-1)$ (Boundary)

$f(0,0) = -\dfrac{1}{25}$ $\quad f(0,-1) = \dfrac{1}{-24}$ $\quad f\left(-\frac{1}{2},\frac{1}{2}\right) = \dfrac{1}{-24.5}$ $\quad f(1,0) = -\dfrac{1}{24}$

$f(-2,-1) = -\dfrac{1}{20}$ $\quad f(1,-1) = -\dfrac{1}{23}$ $\quad f(1,2) = \dfrac{1}{-20}$

$-\dfrac{1}{25}$ Abs. Max Value

$-\dfrac{1}{20}$ Abs. Min. Value

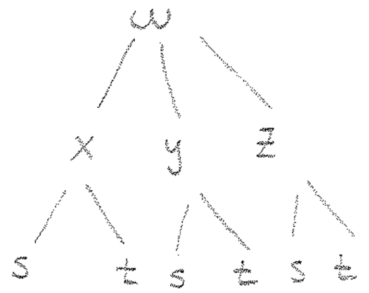
5. (a) (5+6 pts) Find the linear approximation to the function $f(x,y,z) = \sqrt{x^2+1}/y$ at the point $(2,1)$ and use it to estimate the value of $\sqrt{(1.99)^2+1}/1.02$.

Linearization at $(2,1)$ is $L(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$

$f_x = \dfrac{2x}{2\sqrt{x^2+1}\cdot y}$ $\quad f_x(2,1) = \dfrac{2}{\sqrt{5}}$ $\quad L(x,y) = \sqrt{5} + \dfrac{2}{\sqrt{5}}(x-2) - \sqrt{5}(y-1)$

$f_y = \dfrac{-\sqrt{x^2+1}}{y^2}$ $\quad f_y(2,1) = -\sqrt{5}$ $\quad f(1.99, 1.02) \approx L(1.99, 1.02) = \sqrt{5} - \dfrac{0.02}{\sqrt{5}} - 0.02\sqrt{5}$

(b) (pts) Suppose that $w = x^2 + f(y,z)$, $x = t/s$, $y = s-t$ and $z = t^2$. Compute $\partial w/\partial t$ in terms of f_y and f_z.

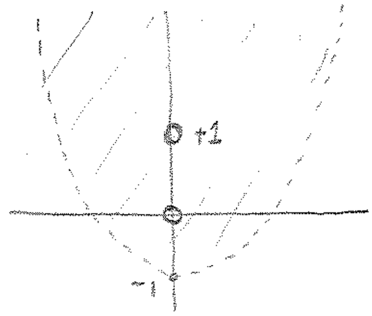


$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t}$$

$$= 2x\cdot\frac{1}{s} + f_y\cdot(-1) + f_z\cdot 2t$$

6.(4+6 pts) Given $f(x,y) = \dfrac{x^3y}{(x^2+y^2)\sqrt{y-x^2+1}} + \dfrac{x^2}{x^2+(y-1)^2}$

(a) Sketch the domain of f



$x^2+y^2 \neq 0 \Rightarrow (0,0)$ is not Dom(f)

$y - x^2 + 1 > 0 \Rightarrow y > x^2 - 1$

$x^2 + (y-1)^2 \neq 0$ $\quad (0,1)$ is not Dom(f)

(b) Show that $\lim\limits_{(x,y)\to(0,1)} f(x,y)$ doesn't exist.

$y = mx + 1$

$$\lim_{x\to 0} \frac{x^3\cdot(mx+1)}{(x^2+(mx+1)^2)\sqrt{mx+1-x^2+1}} + \frac{\cancel{x^2}}{\cancel{x^2}(1+m^2)} = \frac{0}{\sqrt{2}} + \frac{1}{(1+m^2)} = \frac{1}{1+m^2}$$

it changes as m changes. Hence, Limit doesn't exist.

Bonus: (5 pts) Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists.

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+(y-1)^2} = \frac{0}{1} = 0 \quad \text{since it's cont.}$$

$$0 \leq \left|\frac{x^3y}{(x^2+y^2)\sqrt{y-x^2+1}}\right| = \frac{x^2}{(x^2+y^2)}\cdot\frac{|xy|}{\sqrt{y-x^2+1}} < \frac{|xy|}{\sqrt{y-x^2+1}}$$

$\left.\right\}\ \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

by Squeeze Thm

as $(x,y)\to(0,0)$

0

7.(10 pts)Use the method of Lagrange multipliers to find the minimum and the maximum values of $f(x,y) = x^2 - y^2$ subject to $x^2 + 2y^2 = 1$.

Let $g(x,y) = x^2+2y^2$, so our constraint becomes $g(x,y)=1$

$\nabla f = \lambda \nabla g$

(1) $2x = \lambda 2x \Rightarrow 2x(1-\lambda)=0 \quad x=0$ or $\lambda = 1$

(2) $-2y = \lambda 4y$

(3) $x^2+2y^2=1$

$x=0$: $2y^2=1$, $y=\mp\frac{1}{\sqrt{2}}$

$\lambda=1$: $-2y=4y$, $y=0$, $x^2=1 \Rightarrow x=\mp 1$

(If $\lambda=0$, $x=0$, $y=0$ but it doesn't satisfy (3))

$f(0,\frac{1}{\sqrt{2}}) = -\frac{1}{2}$, $f(0,-\frac{1}{\sqrt{2}}) = -\frac{1}{2}$ Minimum Value

$f(1,0) = 1$, $f(-1,0) = 1$ Maximum Value.

8.(5+5 pts)This problem has two unrelated parts.

(a) If the tangent plane to the graph of $f(x,y)$ at (1,2) is given by $z = -3 + 2x - 3y$, then compute the directional derivative $D_u f(1,2)$ in the direciton of $u =< 1, -1 >$.

Normal to the tangent plane is $\langle \hat{f}_x, \hat{f}_y, -1\rangle$, so $\langle \hat{f}_x, \hat{f}_y\rangle = \langle -2, +3\rangle$ at (1,2)

$D_u f(1,2) = \nabla f(1,2)\cdot\frac{u}{|u|} = \langle -2,3\rangle \cdot \frac{\langle 1,-1\rangle}{\sqrt{2}} = \frac{-5}{\sqrt{2}}$.

(b) A bug is flying around a room in which the temperature is given by $T(x,y,z) = x^2e^{yz^2}$. The bug is at the point (1,0,2) and realizes that it's cold. In what direction should it fly to warm up most quickly? What will be the maximum rate of change in its temperature if it goes in that direction?

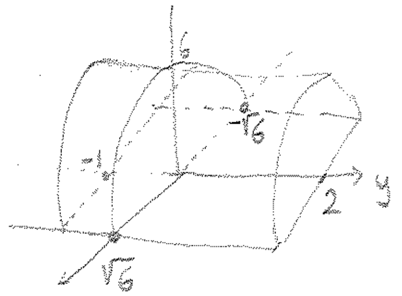
It must fly in $\nabla T(1,0,2)$ direction.

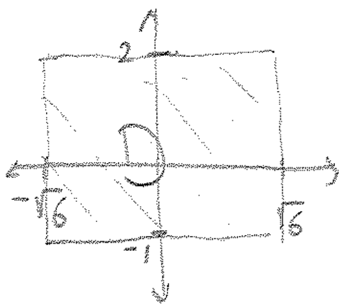
$\nabla T(1,0,2) = \langle T_x, T_y, T_z\rangle\big|_{(1,0,2)} = \langle 2xe^{yz^2}, x^2e^{yz^2}\cdot z^2, x^2e^{yz^2}\cdot 2zy\rangle\big|_{(1,0,2)}$

$= \langle 2, 4, 0\rangle$

Maximum rate of change is equal to $|\nabla T| = \sqrt{20}$

9.(10 pts) Find the volume bounded by the parabolic cylinder $z = 6 - x^2$, the planes $z = 0$, $y = -1$ and $y = 2$ using a double integral.





$$\iint_D [(6-x^2)-0]\,dA = \int_{-\sqrt{6}}^{\sqrt{6}}\int_{-1}^{2} 6-x^2\,dy\,dx$$

$$= \int_{-\sqrt{6}}^{\sqrt{6}}(6-x^2)\,dx \cdot \int_{-1}^{2} dy$$

$$= \left(6x-\frac{x^3}{3}\right)\Bigg|_{-\sqrt{6}}^{\sqrt{6}} \cdot \; y\Bigg|_{-1}^{2}$$

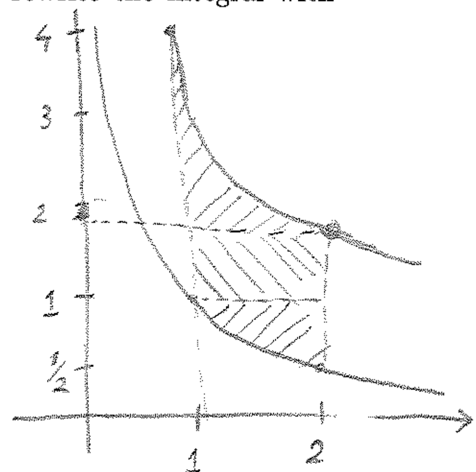
$$= 2\left(6\sqrt{6} - \frac{6\sqrt{6}}{3}\right)\cdot(2-(-1))$$

10.(10 pts) This problem has two unrelated parts. DO NOT EVALUATE.

(a) Sketch the region of integration for the following integral and rewrite the integral with reverse iteration.

$$\int_1^2\int_{\frac{1}{x}}^{\frac{4}{x}} f(x,y)dydx$$

$$= \int_{1/2}^{1}\int_{1/y}^{2} f(x,y)\,dx\,dy + \int_{1}^{2}\int_{1}^{2} f(x,y)\,dx\,dy + \int_{2}^{4}\int_{1}^{4/y} f(x,y)\,dx\,dy$$



(b) Write an iterated integral for $\iint_D f(x,y)dA$ where R is bounded by the curves $x = y^2 - 1$ and $y = x - 1$

$y^2 - 1 = y+1 \Rightarrow y^2 - y - 2 = 0$

$(y-2)(y+1) = 0 \Rightarrow y = 2 \quad x = 3$

$y = -1 \quad x = 0$

$$\int_{-1}^{2}\int_{y^2-1}^{y+1} f(x,y)\,dx\,dy$$