METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES
MIDTERM 1

Code : *MAT 120*	Last Name:		
Acad.Year: *2012-2013*	Name :	Student No.:	
Semester : *Spring*	Department:	Section:	
Date : *23.03.2013*	Signature :		
Time : *9:40*	7 QUESTIONS ON 4 PAGES		
Duration : *100 min*	TOTAL 100 POINTS		

1. (20)	2. (16)	3. (9)	4. (15)	5. (10)	6. (20)	7. (10)	

1. *(8+2+10pts)* In parts **A-B**, let $\mathbf{u} = \langle 1,\ 1,\ -3 \rangle$ and $\mathbf{v} = \langle 2,\ -1,\ 5 \rangle$.

(A) Find vectors $\mathbf{u}_1, \mathbf{u}_2$ so that $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ with $\mathbf{u}_1 \parallel \mathbf{v}$ and $\mathbf{u}_2 \perp \mathbf{v}$.

(i.e. $\mathbf{u}_1$ is parallel to $\mathbf{v}$; and $\mathbf{u}_2$ is perpendicular to $\mathbf{v}$.)

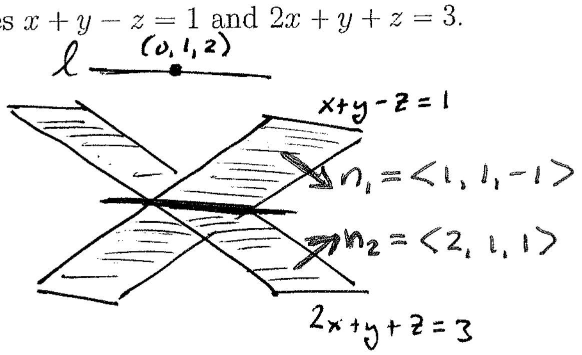
- $$\bar{u}_1 = \mathrm{Proj}_{\bar{v}}\, \bar{u} = \frac{\bar{u}\cdot\bar{v}}{\bar{v}\cdot\bar{v}}\,\bar{v} = \frac{\langle 1, 1, -3\rangle \cdot \langle 2, -1, 5\rangle}{\langle 2, -1, 5\rangle \cdot \langle 2, -1, 5\rangle} \langle 2, -1, 5\rangle$$
$$= -\frac{14}{30}\langle 2, -1, 5\rangle = \boxed{-\tfrac{7}{15}\langle 2, -1, 5\rangle}$$

- $$\bar{u}_2 = \bar{u} - \bar{u}_1 = \langle 1, 1, -3\rangle - \left(-\tfrac{7}{15}\langle 2, -1, 5\rangle\right)$$
$$= \boxed{\tfrac{1}{15}\langle 29, 8, -10\rangle}$$

(B) Verify that the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are perpendicular.

$$\bar{u}_1 \cdot \bar{u}_2 = -\tfrac{7}{15^2}\left(2\cdot 29 - 8 - 10\cdot 5\right) = 0. \quad \text{So } \bar{u}_1 \perp \bar{u}_2.$$

(C) Write an equation for the line through $(0,\ 1,\ 2)$ which is parallel to the intersection of the planes $x + y - z = 1$ and $2x + y + z = 3$.



Direction of intersection line

$$\langle 1, 1, -1\rangle \times \langle 2, 1, 1\rangle = \langle 2, -3, -1\rangle$$

$$\boxed{\begin{array}{l} \ell \text{ is line } \bar{r}(t) = \langle 0, 1, 2\rangle + \langle 2, -3, 1\rangle t \\[1em] \begin{cases} x(t) = 2t \\ y(t) = 1 - 3t \\ z(t) = 2 - t \end{cases} \end{array}}$$

2. *(4×4pts)* In parts **A-B** below, let $\mathbf{r}(t) = \langle t,\ t^2,\ t^3 \rangle$.

(A) Compute the velocity, $\mathbf{r}'(t)$.

$$\boxed{\bar{r}'(t) = \langle 1,\ 2t,\ 3t^2 \rangle}$$

(B) Compute the definite integral, $\int_0^2 \mathbf{r}'(t)\,dt$.

$$\int_0^2 \bar{r}'(t)\,dt = \left\langle \int_0^2 1\,dt,\ \int_0^2 2t\,dt,\ \int_0^2 3t^2\,dt \right\rangle$$

$$= \left\langle t\,\Big|_0^2,\ t^2\,\Big|_0^2,\ t^3\,\Big|_0^2 \right\rangle = \boxed{\langle 2,\ 4,\ 8 \rangle}$$

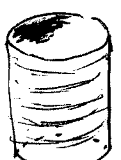
(Note: You could also use Fundamental Theorem of Calculus.)

Parts **C-D** below involve the hyperbolic paraboloid $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$.

(C) Give a vector function $\mathbf{s}(t) = \langle x(t),\ y(t),\ z(t) \rangle$ representing the curve of intersection of $z = x^2 - y^2$ and $x^2 + y^2 = 1$.

On cylinder $x^2+y^2=1$,

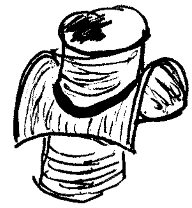
$$\rightarrow \begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}$$



On hyperbolic paraboloid

$$z = x^2 - y^2$$

$$\rightarrow z(t) = \cos^2 t - \sin^2 t$$



$$\boxed{\bar{s}(t) = \langle \cos t,\ \sin t,\ \cos^2 t - \sin^2 t \rangle = \langle \cos t,\ \sin t,\ \cos 2t \rangle} \quad (0 \le t \le 2\pi)$$

(D) Use your answer from **C** to compute a vector tangent to the intersection of $z = x^2 - y^2$ and $x^2 + y^2 = 1$ at the point $(0, -1, -1)$.

$$\bar{s}'(t) = \langle -\sin t,\ \cos t,\ -2\sin 2t \rangle$$

at the point $(0,-1,-1)$ $\quad \begin{matrix} x=0 \\ y=-1 \end{matrix}\ \}$ $\quad t = 3\pi/2$ $\quad (0,-1)$

$$\bar{s}'(3\pi/2) = \langle -\sin 3\pi/2,\ \cos 3\pi/2,\ -2\sin 3\pi \rangle$$

$$= \boxed{\langle 1,\ 0,\ 0 \rangle}$$

3. *(9pts)* Show that the following limit does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^3+y^2}$

Consider limits along the following paths.

- $(x,y) \to (0,0)$ by $\begin{cases} x=0 \\ y \to 0 \end{cases}$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^3+y^2} = \lim_{y\to 0} \frac{0}{y^3} = 0$

- $(x,y) \to (0,0)$ by $\begin{cases} x=y \\ y \to 0 \end{cases}$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^3+y^2} = \lim_{y\to 0} \frac{y^2}{y^3+y^2} = \lim_{y\to 0} \frac{1}{y+1} = 1$

These two paths to $(0,0)$ give different limits, so $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^3+y^2}$ does not exist.

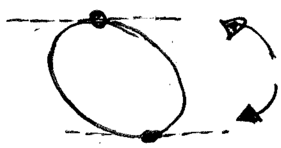
(This should have been the last page)

4. *(5+10pts)* In parts A-B below, let $F(x,y) = x^2 + xy + y^2 + y$.

(A) Find the partial derivatives of $F(x,y)$.

$$\boxed{\begin{aligned} \partial F/\partial x &= 2x + y \\ \partial F/\partial y &= x + 2y + 1 \end{aligned}}$$

(B) Determine the points (x,y) where the curve $F(x,y) = 0$ has horizontal tangent lines.



Horizontal tangent line $\Longleftrightarrow$ $dy/dx\,(a,b) = 0$.

$\Longleftrightarrow$ Normal vector to level curve is $\langle 0, k\rangle$

$\Longleftrightarrow$ $\partial F/\partial x = 0$

$2x + y = 0$

$y = -2x$

(Note: We could also have used implicit diff:

$0 = dy/dx = -\dfrac{F_x}{F_y} \Longleftrightarrow F_x = 0$)

Want points on $x^2 + xy + y^2 + y = 0$ with $y = -2x$

$x^2 - 2x^2 + 4x^2 - 2x = 0$

$x(3x-2) = 0$

$x = 0 \rightarrow y = 0$

$x = 2/3 \rightarrow y = -4/3$

$\boxed{(0,0) \text{ and } (2/3, -4/3)}$

5. *(10pts)* Consider the function $f(x,y) = x\,e^{xy}$ at the point $(\sqrt{2}, 0)$. Find all direction vectors $\mathbf{u}$ so that $D_{\mathbf{u}} f\,(\sqrt{2}, 0) = 2$.

$\nabla f = \langle f_x, f_y \rangle = \langle e^{xy} + xye^{xy}, \; x^2 e^{xy} \rangle$

$\hookrightarrow \nabla f(\sqrt{2}, 0) = \langle e^0 + 0, \; 2e^0 \rangle = \langle 1, 2 \rangle$

If $\bar{u} = \langle a, b \rangle$ with $a^2 + b^2 = 1$ then

$D_{\bar{u}} f(\sqrt{2}, 0) = \bar{u} \cdot \nabla f(\sqrt{2}, 0)$

$= \langle a, b \rangle \cdot \langle 1, 2 \rangle$

$= a + 2b$

Want: $a + 2b = 2 \rightsquigarrow a = 2 - 2b$

$a^2 + b^2 = 1 \rightsquigarrow (2-2b)^2 + b^2 = 1$

$4b^2 - 8b + 4 + b^2 = 1$

$5b^2 - 8b + 3 = 1$

$b = \dfrac{8 \pm \sqrt{64-60}}{10} = \dfrac{4 \pm 1}{5}$

$(a = 2 - 2b)$

$b = 1 \longrightarrow a = 0$

or

$b = 3/5 \longrightarrow a = 4/5$

$\boxed{\langle 0, 1 \rangle \text{ and } \langle 4/5, 3/5 \rangle}$

6. *(10+10pts)* Give equations for the following tangent planes.

(A) The tangent plane to $z = \underbrace{xy + 2x + 3y}_{f(x,y)}$ at $x = 1,\ y = 2$.

$f(1,2) = 10$

$f_x(x,y) = y + 2 \rightarrow f_x(1,2) = 4$

$f_y(x,y) = x + 3 \rightarrow f_y(1,2) = 4$

Tangent plane is
$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$

$$\boxed{z = 4(x-1) + 4(y-2) + 10}$$

(B) The tangent plane to $\underbrace{x^2 + y^2 - z^2}_{F(x,y,z)} = 1$ at the point (1, 2, 2).

$F_x = 2x \rightarrow F_x(1,2,2) = 2$

$F_y = 2y \rightarrow F_y(1,2,2) = 4$

$F_z = -2z \rightarrow F_z(1,2,2) = -4$

Tangent plane is

$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$

$$\boxed{2(x-1) + 4(y-2) + (-4)(z-2) = 0}$$

7. *(10pts)* Let $F(x, y, z) = zx + \sin(2yz)$ where $\begin{cases} x(s,t) = 2s + 3st, \\ y(s,t) = 3t + 2st, \\ z(x,y,t) = x + yt. \end{cases}$

$x(1,2) = 8$

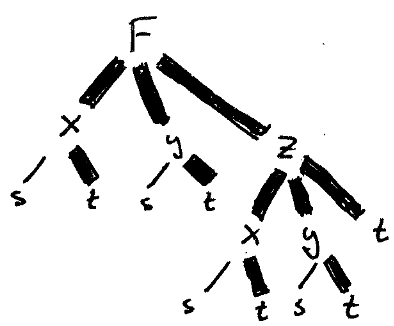
$y(1,2) = 10$

$z(8,10,2) = 28$

used below.

Use the chain rule to compute $\frac{\partial F}{\partial t}$ at the point $s = 1,\ t = 2$.

(computations which do not use the chain rule will receive no credit)



$F_x = z \rightarrow F_x(1,2) = 28$

$F_y = 2z\cos(2yz) \rightarrow F_y(1,2) = 48\cos(560)$

$F_z = x + 2y\cos(2yz) \rightarrow F_z(1,2) = 8 + 20\cos(560)$

$x_t = 3s \rightarrow x_t(1,2) = 3$

$y_t = 3 + 2s \rightarrow y_t(1,2) = 5$

$z_x = 1 \rightarrow z_x(1,2) = 1$

$z_y = t \rightarrow z_y(1,2) = 2.$

$z_t = z_x x_t + z_y y_t + y$

$z_t(1,2) = 1\cdot 3 + 2\cdot 5 + 10$

$= 23$

$F_t = F_x x_t + F_y y_t + F_z z_t$

$$= \boxed{84 + 48\cos(560)\cdot 5 + (8 + 20\cos(560))\cdot 23}$$