METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES FINAL EXAM	
Code : *MAT 120* Acad.Year: *2013-2014* Semester : *FALL* Date : *20.01.2014* Time : *16:00* Duration : *120 min*	Last Name: Name : Solutions Student # : Signature :
	7 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1. (12)	2. (12)	3. (15)	4. (16)	5. (16)	6. (15)	7. (14)		

Please draw a box around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(3×4=12pts)* The following parts all deal with properties of the surface $z = 4x^2y + 2y^3 - 1$ at the point $(2, 1, 17)$.

(a) Write the equation for the tangent plane to $z = 4x^2y + 2y^3 - 1$ at the point $(2, 1, 17)$.

$$\left.\begin{aligned} dz/dx &= 8xy \\ dz/dy &= 4x^2 + 6y^2 \end{aligned}\right\} @ {x=2 \atop y=1} \left\{\begin{aligned} z_x(2,1) &= 16 \\ z_y(2,1) &= 22 \end{aligned}\right.$$

$$\boxed{\text{Tangent Plane: } z = 16(x-2) + 22(y-1) + 17 \quad \text{-or-} \quad 16x + 22y - z = 37}$$

(b) Compute the directional derivative of $z = 4x^2y + 2y^3 - 1$ when $x = 2$ and $y = 1$ in the direction of $\mathbf{u} = \langle 1, 0\rangle$.

$$D_{\langle 1,0\rangle} z(2,1) = \langle 1,0\rangle \cdot \langle z_x(2,1), z_y(2,1)\rangle$$
$$= \langle 1,0\rangle \cdot \langle 16, 22\rangle$$
$$= \boxed{16}$$

(Note: $D_{\langle 1,0\rangle} f = \partial/\partial x\, f$)

(c) Compute the directional derivative of $z = 4x^2y + 2y^3 - 1$ when $x = 2$ and $y = 1$ in the direction of $\mathbf{u} = \frac{1}{\sqrt{5}}\langle 1, 2\rangle$.

$$D_{1/\sqrt{5}\langle 1,2\rangle} z(2,1) = 1/\sqrt{5} \cdot \langle 1,2\rangle \cdot \langle z_x(2,1), z_y(2,1)\rangle$$
$$= 1/\sqrt{5}\, \langle 1,2\rangle \cdot \langle 16, 22\rangle$$
$$= 1/\sqrt{5}\,(16+44) = \boxed{60/\sqrt{5}}$$

(d) Write the vector equation of a line (in 3D) perpendicular to the tangent plane from (a), through the point $(2, 1, 17)$.

$$\boxed{r(t) = \langle 16, 22, -1\rangle t + \langle 2, 1, 17\rangle}$$

2. *(3+3+6=12pts)* The following parts are about triple integrals.

(a) Write a triple integral computing $\iiint_E x\,dV$ where E is the region inside of $y = x^2$, $x = y^2$, $z = 0$, and $z = x + y$.

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=\sqrt{y}} \int_{z=0}^{z=x+y} x \; dz\, dx\, dy$$

(b) Write another triple integral computing $\iiint_E x\,dV$ using a different order of integration than **(a)**.

$$\int_{x=0}^{x=1} \int_{y=x^2}^{y=\sqrt{x}} \int_{z=0}^{z=x+y} x \; dz\, dy\, dx$$

(Note: If z is not integrated first then this is very messy.)

(c) Solve either the triple integral from part **(a)** or from part **(b)**.

$$\int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} x \; dz\, dy\, dx = \int_0^1 \int_{x^2}^{\sqrt{x}} xz \Big|_0^{x+y} dy\, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} x(x+y) \; dy\, dx$$

$$= \int_0^1 x^2 y + \tfrac{1}{2} x y^2 \Big|_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 x^{5/2} + \tfrac{1}{2} x^2 - x^4 - \tfrac{1}{2} x^5 \; dx$$

$$= 2/7\, x^{7/2} + \tfrac{1}{6} x^3 - 1/5\, x^5 - \tfrac{1}{12} x^6 \Big|_0^1$$

$$= \boxed{2/7 + 1/6 - 1/5 - 1/12 = 71/420}$$

3. *(8+8=16pts)* Compute the following line integrals using the definitions.

(a) $\int_C xy + y^3 \; ds$ where C is $\{\mathbf{r}(t) = \langle t^2,\, 2t\rangle,\ 0 \leqslant t \leqslant 3\}$

$$\int_0^3 (\underbrace{t^2\cdot 2t}_{x\cdot y} + \underbrace{(2t)^3}_{+\ y^3})\underbrace{\sqrt{(2t)^2+2^2}}_{ds}\,dt = \int_0^3 (2t^3+8t^3)\sqrt{4t^2+4}\,dt$$

$$= \int_0^3 20\,t^3\sqrt{t^2+1}\,dt$$

$$\left.\begin{array}{l} u = t^2+1 \\ du = 2t\,dt \\ u-1 = t^2 \end{array}\right\} = \int_1^{10} 10\,(u-1)\,u^{1/2}\,du$$

$$= 10\left(\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right)\Big|_1^{10}$$

$$= \boxed{10\left(\tfrac{2}{5}10^{5/2} - \tfrac{2}{3}10^{3/2} - \tfrac{2}{5} + \tfrac{2}{3}\right)}$$

(b) $\int_C xy\; dx + y^3\; dy$ where C is $\{\mathbf{r}(t) = \langle t^2,\, 2t\rangle,\ 0 \leqslant t \leqslant 3\}$

$$\int_0^3 \underbrace{t^2}_{x}\cdot\underbrace{2t}_{y}\cdot\underbrace{2t}_{dx} + \underbrace{(2t)^3}_{y^3}\cdot\underbrace{2}_{dy}\;dt = \int_0^3 4t^4 + 16t^3\;dt$$

$$= \tfrac{4}{5}t^5 + 4t^4\Big|_0^3$$

$$= \boxed{\tfrac{4}{5}3^5 + 4\cdot 3^4 = \tfrac{972}{5} + 324}$$

4. *(5×3=15pts)* The vector field $\mathbf{F} = \langle e^{xy} + xye^{xy},\ x^2e^{xy} + y\rangle$ is conservative.

(a) Show that $\mathbf{F}$ is conservative.

$$\tfrac{\partial}{\partial y}\left(e^{xy} + xye^{xy}\right) = xe^{xy} + \left(xe^{xy} + x^2ye^{xy}\right)$$

$$= 2xe^{xy} + x^2ye^{xy}$$

These are equal, so $\underline{F}$ is conservative.

$$\tfrac{\partial}{\partial x}\left(x^2e^{xy} + y\right) = 2xe^{xy} + x^2ye^{xy} + 0$$

(b) Find ~~the~~ a potential function for $\mathbf{F}$.

$$\text{y-terms of } f = \int x^2e^{xy} + y\;dy = \frac{x^2}{x}e^{xy} + \tfrac{1}{2}y^2 \qquad \tfrac{\partial}{\partial x}$$

$$\text{new x-terms of } f = \int e^{xy} + xye^{xy} - \left(e^{xy} + xye^{xy} + 0\right)\;dx$$

$$= \int 0\;dx$$

$$\boxed{f = xe^{xy} + \tfrac{1}{2}y^2}$$

(c) Use your answer from **(b)** to compute the line integral

$\int_C \mathbf{F} \bullet d\mathbf{r}$ where C is $\{\mathbf{r}(t) = \langle \sin(\pi t^2),\, \arctan(t)\rangle,\ 0 \leqslant t \leqslant 1\}$

start point $= (\sin(0), \arctan 0) = (0, 0)$

end point $= (\sin(\pi), \arctan 1) = (0, \tfrac{\pi}{4})$

$$\int_C \underline{F}\cdot d\underline{r} = f(\text{end}) - f(\text{start}) = 0e^{0\cdot\pi/4} + \tfrac{1}{2}(\tfrac{\pi}{4})^2 - (0+0) = \boxed{\tfrac{\pi^2}{32}}$$

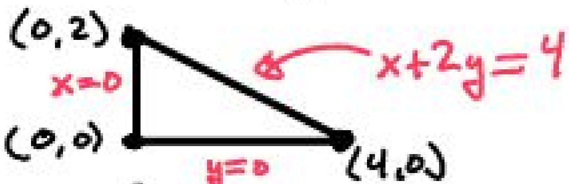
Green's Thm: $\int_C P\,dx + Q\,dy = \iint_{\text{inside } C} Q_x - P_y \; dx\,dy$

5. *(4×4=16pts)* The following parts are about Green's Theorem. For parts **a-c**, use Green's Theorem to convert the circulation integrals to double integrals. DO NOT INTEGRATE. C is the loop going counter-clockwise around the triangle with vertices $(0,0)$, $(0,2)$, $(4,0)$.

(a) $\oint_C (2y + e^{\sqrt{x}})\,dx + \left(3x^2 - \operatorname{arcsec}(y^3+1)\right)\,dy$

$$\left.\begin{aligned} \partial/\partial y\,(2y + e^{\sqrt{x}}) &= 2 \\ \partial/\partial x\,(3x^2 - \operatorname{arcsec}(y^3+1)) &= 6x \end{aligned}\right\} \quad \iint_{\triangle} 6x - 2 \; dx\,dy$$

$$= \boxed{\int_{y=0}^{y=2} \int_{x=0}^{x=4-2y} 6x - 2 \; dx\,dy}$$



(b) $\oint_C e^{\sqrt{x}}\,dx + \operatorname{arcsec}(y^3+1)\,dy$

$$\left.\begin{aligned} \partial/\partial y\; e^{\sqrt{x}} &= 0 \\ \partial/\partial x\; \operatorname{arcsec}(y^3+1) &= 0 \end{aligned}\right\} \quad \iint_{\triangle} 0 - 0 \; dx\,dy = \boxed{0}$$

Note: This is circulation of a conservative vector field!!!

(c) $\oint_C 2y\,dx + 3x^2\,dy$

$$\left.\begin{aligned} \partial/\partial y\; 2y &= 2 \\ \partial/\partial x\; 3x^2 &= 6x \end{aligned}\right\} \quad \iint_{\triangle} 6x - 2 \; dx\,dy$$

$$= \boxed{\int_{y=0}^{y=2} \int_{x=0}^{x=4-2y} 6x - 2 \quad dx\,dy}$$

(d) Solve the double integral from **(a)**. INTEGRATE.

$$\int_0^2 \int_0^{4-2y} 6x - 2y \; dx\,dy = \int_0^2 3x^2 - 2x \;\Big|_0^{4-2y} dy$$

$$= \int_0^2 3(4-2y)^2 - 2(4-2y) \; dy$$

$$= \int_0^2 12y^2 - 44y + 40 \; dy$$

$$= 4y^3 - 22y^2 + 40y \;\Big|_0^2$$

$$= 32 - 88 + 80 = \boxed{24}$$

$(4+4+7=15pts)$

6. $(\cancel{3+3+3+6=15pts})$ The following parts are about convergence/divergence of series. State which convergence tests you use and give all details.

For parts **a**-**b**, determine whether the series are convergent or divergent. In part **c**, find the interval of convergence.

(a) $\displaystyle\sum_{n=5}^{\infty} \frac{n+1}{\sqrt{n^3-7n+12}}$ Limit comparison with $\displaystyle\sum_1 \frac{n}{\sqrt{n^3}} = \sum_1 \frac{1}{n^{1/2}}$

$$\lim \frac{\frac{n+1}{\sqrt{n^3-7n+12}}}{\frac{n}{\sqrt{n^3}}} = \lim \frac{\frac{n+1}{n}}{\sqrt{\frac{n^3-7n+12}{n^3}}} = \lim \frac{1+1/n}{\sqrt{1-7/n^2+12/n^3}} = 1 \neq 0,\ \neq \infty$$

Since $\sum_1 1/n^{1/2}$ is a divergent p-series $(p=\frac{1}{2}\leq 1)$, the series is $\boxed{\text{Divergent}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \cos(\pi n)$

$\sum_1 \cos(\pi n) = -1+1-1+1-1+1 \cdots$ Note: $\lim \cos(\pi n) = \lim (-1)^n \neq 0$ (In fact, it does not exist!)

This series is $\boxed{\text{Divergent}}$ by the Test for Divergence (a.k.a. the n^{th} term test)

(c) Find the interval of convergence of the power series $\displaystyle\sum_{n=3}^{\infty} (-1)^n \frac{(2x-5)^n}{3^{2n}\sqrt{n}}$ $\displaystyle\sum_1 (-1)^n \frac{2^n(x-5/2)^n}{3^{2n}\sqrt{n}}$

The interval has center at $2x-5=0$
$x = 5/2$

Find the radius of convergence w/ Ratio Test

Convergent if

$$1 > \lim \left| \frac{(-1)^{n+1}\frac{(2x-5)^{n+1}}{3^{2n+2}\sqrt{n+1}}}{(-1)^n \frac{(2x-5)^n}{3^{2n}\sqrt{n}}} \right| = \lim \frac{|2x-5|}{3^2\sqrt{\frac{n+1}{n}}} = \frac{|2x-5|}{3^2}$$

Radius of convergence

so $|2x-5| < 3^2$ or $|x-5/2| < 3^2/2$

If $x - 5/2 = -3^2/2$ then the power series is

$$\sum_1 (-1)^n \frac{2^n(-3^2/2)^n}{3^{2n}\sqrt{n}} = \sum_1 1/\sqrt{n}$$ a divergent p-series $(p=1/2 \leq 1)$

If $x-5/2 = 3^2/2$ then the power series is

$$\sum_1 (-1)^n \frac{2^n(3^2/2)^n}{3^{2n}\sqrt{n}} = \sum_1 (-1)^n 1/\sqrt{n}$$ a convergent alternating series ($1/\sqrt{n}$ is decreasing)

Interval of convergence $= \boxed{(5/2 - 3^2/2,\ 5/2 + 3^2/2] = (-2, 7]}$

7. *(7+7=14pts)* The following parts are about finding/manipulating power series.

(a) Use Taylor's Theorem to compute the series representation of $\ln(x)$ around $a = 3$.

$$f(x) = \ln x \qquad f(3) = \ln 3$$
$$f'(x) = 1/x \qquad f'(3) = 1/3$$
$$f''(x) = -1/x^2 \qquad f''(3) = -1/3^2$$
$$f'''(x) = 2/x^3 \qquad f'''(3) = 2/3^3$$
$$f^{(4)}(x) = -2\cdot 3/x^4 \qquad f^{(4)}(3) = -2\cdot 3/3^4$$

Taylor's Thm:
$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

pattern: $f^{(n)}(3) = (-1)^{n+1}\dfrac{(n-1)!}{3^n}$ (for $n > 1$)

Thus $$\ln x = \ln 3 + \sum_{n=1}^{\infty} (-1)^{n+1}\frac{(n-1)!}{3^n\, n!}(x-3)^n = \boxed{\ln 3 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-3)^n}{3^n\cdot n}}$$

(b) Using manipulations of power series, show that $\dfrac{d}{dx}e^{x^2} = 2x\,e^{x^2}$.
(i.e. Write the power series of e^{x^2} and $2x\,e^{x^2}$. Show that the derivative of the first power series equals the second.)

Recall: $$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Thus $$e^{x^2} = \sum_{n=0}^{\infty}\frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$$

also $$\frac{d}{dx}e^{x^2} = \sum_{n=0}^{\infty}\frac{d}{dx}\frac{x^{2n}}{n!} = \sum_{n=1}^{\infty}\frac{2n\,x^{2n-1}}{n!}$$
$$= \sum_{n=1}^{\infty}\frac{2x^{2n-1}}{(n-1)!} = \frac{2x^{2-1}}{0!} + \frac{2x^{4-1}}{1!} + \cdots$$

REINDEX $$= \sum_{n=0}^{\infty}\frac{2x^{2n+1}}{n!} = \frac{2x^{0+1}}{0!} + \frac{2x^{2+1}}{1!} + \cdots$$

OK

similarly $$2xe^{x^2} = 2x\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$$
$$= \sum_{n=0}^{\infty}\frac{2x^{2n+1}}{n!}$$

Yay! These are equal!!!