TEC In Module 3.3 you can practice using information about f', f'', and asymptotes to determine the shape of the graph of f.

at 6, so there is no minimum or maximum there. (The Second Derivative Test could be used at 4 but not at 0 or 6 since f'' does not exist at either of these numbers.)

Looking at the expression for $f''(x)$ and noting that $x^{4/3} \geqslant 0$ for all x, we have $f''(x) < 0$ for $x < 0$ and for $0 < x < 6$ and $f''(x) > 0$ for $x > 6$. So f is concave downward on $(-\infty, 0)$ and $(0, 6)$ and concave upward on $(6, \infty)$, and the only inflection point is $(6, 0)$. The graph is sketched in Figure 12. Note that the curve has vertical tangents at $(0, 0)$ and $(6, 0)$ because $|f'(x)| \to \infty$ as $x \to 0$ and as $x \to 6$.

Try reproducing the graph in Figure 12 with a graphing calculator or computer. Some machines produce the complete graph, some produce only the portion to the right of the y-axis, and some produce only the portion between $x = 0$ and $x = 6$. For an explanation and cure, see Example 7 in Appendix G. An equivalent expression that gives the correct graph is

$$y = (x^2)^{1/3} \cdot \frac{6 - x}{|6 - x|} |6 - x|^{1/3}$$

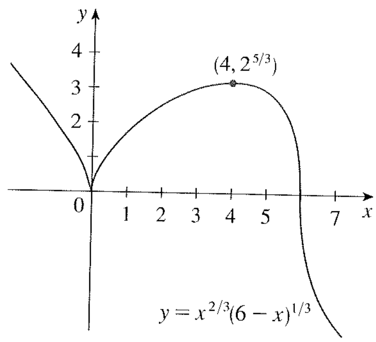


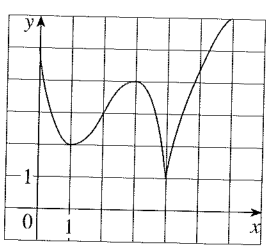
FIGURE 12

3.3 Exercises

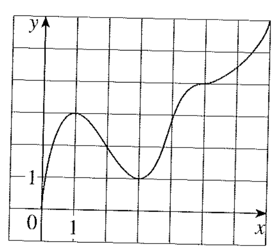
1–2 Use the given graph of f to find the following.

(a) The open intervals on which f is increasing.
(b) The open intervals on which f is decreasing.
(c) The open intervals on which f is concave upward.
(d) The open intervals on which f is concave downward.
(e) The coordinates of the points of inflection.

1.



2.

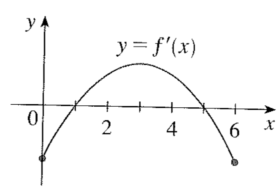


3. Suppose you are given a formula for a function f.
(a) How do you determine where f is increasing or decreasing?
(b) How do you determine where the graph of f is concave upward or concave downward?
(c) How do you locate inflection points?

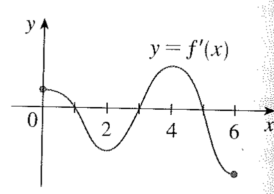
4. (a) State the First Derivative Test.
(b) State the Second Derivative Test. Under what circumstances is it inconclusive? What do you do if it fails?

5–6 The graph of the *derivative* f' of a function f is shown.
(a) On what intervals is f increasing or decreasing?
(b) At what values of x does f have a local maximum or minimum?

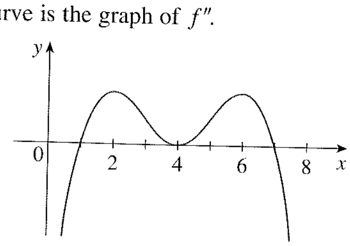
5.



6.

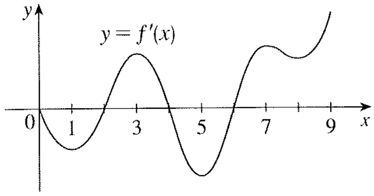


7. In each part state the x-coordinates of the inflection points of f. Give reasons for your answers.
(a) The curve is the graph of f.
(b) The curve is the graph of f'.
(c) The curve is the graph of f''.



8. The graph of the first derivative f' of a function f is shown.
(a) On what intervals is f increasing? Explain.

(b) At what values of x does f have a local maximum or minimum? Explain.
(c) On what intervals is f concave upward or concave downward? Explain.
(d) What are the x-coordinates of the inflection points of f? Why?



9–14
(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum values of f.
(c) Find the intervals of concavity and the inflection points.

9. $f(x) = 2x^3 + 3x^2 - 36x$

10. $f(x) = 4x^3 + 3x^2 - 6x + 1$

11. $f(x) = x^4 - 2x^2 + 3$ **12.** $f(x) = \dfrac{x^2}{x^2 + 3}$

13. $f(x) = \sin x + \cos x, \quad 0 \leqslant x \leqslant 2\pi$

14. $f(x) = \cos^2 x - 2\sin x, \quad 0 \leqslant x \leqslant 2\pi$

15–17 Find the local maximum and minimum values of f using both the First and Second Derivative Tests. Which method do you prefer?

15. $f(x) = x^5 - 5x + 3$ **16.** $f(x) = \dfrac{x^2}{x - 1}$

17. $f(x) = \sqrt{x} - \sqrt[4]{x}$

18. (a) Find the critical numbers of $f(x) = x^4(x - 1)^3$.
(b) What does the Second Derivative Test tell you about the behavior of f at these critical numbers?
(c) What does the First Derivative Test tell you?

19. Suppose f'' is continuous on $(-\infty, \infty)$.
(a) If $f'(2) = 0$ and $f''(2) = -5$, what can you say about f?
(b) If $f'(6) = 0$ and $f''(6) = 0$, what can you say about f?

20–25 Sketch the graph of a function that satisfies all of the given conditions.

20. Vertical asymptote $x = 0$, $f'(x) > 0$ if $x < -2$,
$f'(x) < 0$ if $x > -2$ $(x \neq 0)$,
$f''(x) < 0$ if $x < 0$, $f''(x) > 0$ if $x > 0$

21. $f'(0) = f'(2) = f'(4) = 0$,
$f'(x) > 0$ if $x < 0$ or $2 < x < 4$,
$f'(x) < 0$ if $0 < x < 2$ or $x > 4$,
$f''(x) > 0$ if $1 < x < 3$, $f''(x) < 0$ if $x < 1$ or $x > 3$

22. $f'(1) = f'(-1) = 0$, $f'(x) < 0$ if $|x| < 1$,
$f'(x) > 0$ if $1 < |x| < 2$, $f'(x) = -1$ if $|x| > 2$,
$f''(x) < 0$ if $-2 < x < 0$, inflection point $(0, 1)$

23. $f'(x) > 0$ if $|x| < 2$, $f'(x) < 0$ if $|x| > 2$,
$f'(-2) = 0$, $\lim_{x \to 2} |f'(x)| = \infty$, $f''(x) > 0$ if $x \neq 2$

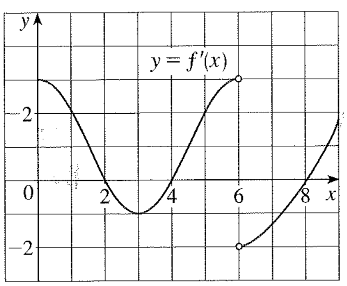
24. $f(0) = f'(0) = f'(2) = f'(4) = f'(6) = 0$,
$f'(x) > 0$ if $0 < x < 2$ or $4 < x < 6$,
$f'(x) < 0$ if $2 < x < 4$ or $x > 6$,
$f''(x) > 0$ if $0 < x < 1$ or $3 < x < 5$,
$f''(x) < 0$ if $1 < x < 3$ or $x > 5$, $f(-x) = f(x)$

25. $f'(x) < 0$ and $f''(x) < 0$ for all x

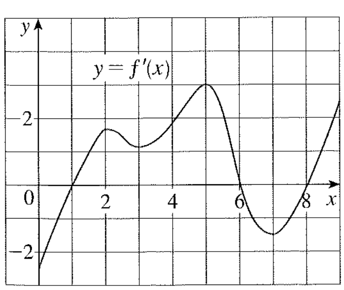
26. Suppose $f(3) = 2$, $f'(3) = \frac{1}{2}$, and $f'(x) > 0$ and $f''(x) < 0$ for all x.
(a) Sketch a possible graph for f.
(b) How many solutions does the equation $f(x) = 0$ have? Why?
(c) Is it possible that $f'(2) = \frac{1}{3}$? Why?

27–28 The graph of the derivative f' of a continuous function f is shown.
(a) On what intervals is f increasing? Decreasing?
(b) At what values of x does f have a local maximum? Local minimum?
(c) On what intervals is f concave upward? Concave downward?
(d) State the x-coordinate(s) of the point(s) of inflection.
(e) Assuming that $f(0) = 0$, sketch a graph of f.

27.



28.



29–40

(a) Find the intervals of increase or decrease.
(b) Find the local maximum and minimum values.
(c) Find the intervals of concavity and the inflection points.
(d) Use the information from parts (a)–(c) to sketch the graph. Check your work with a graphing device if you have one.

29. $f(x) = 2x^3 - 3x^2 - 12x$ **30.** $f(x) = 2 + 3x - x^3$

31. $f(x) = 2 + 2x^2 - x^4$ **32.** $g(x) = 200 + 8x^3 + x^4$

33. $h(x) = (x + 1)^5 - 5x - 2$ **34.** $h(x) = 5x^3 - 3x^5$

35. $F(x) = x\sqrt{6 - x}$ **36.** $G(x) = 5x^{2/3} - 2x^{5/3}$

37. $C(x) = x^{1/3}(x + 4)$ **38.** $G(x) = x - 4\sqrt{x}$

39. $f(\theta) = 2\cos\theta + \cos^2\theta, \quad 0 \leqslant \theta \leqslant 2\pi$

40. $S(x) = x - \sin x, \quad 0 \leqslant x \leqslant 4\pi$

41. Suppose the derivative of a function f is $f'(x) = (x + 1)^2(x - 3)^5(x - 6)^4$. On what interval is f increasing?

42. Use the methods of this section to sketch the curve $y = x^3 - 3a^2x + 2a^3$, where a is a positive constant. What do the members of this family of curves have in common? How do they differ from each other?

43–44

(a) Use a graph of f to estimate the maximum and minimum values. Then find the exact values.
(b) Estimate the value of x at which f increases most rapidly. Then find the exact value.

43. $f(x) = \dfrac{x + 1}{\sqrt{x^2 + 1}}$

44. $f(x) = x + 2\cos x, \quad 0 \leqslant x \leqslant 2\pi$

45–46

(a) Use a graph of f to give a rough estimate of the intervals of concavity and the coordinates of the points of inflection.
(b) Use a graph of f'' to give better estimates.

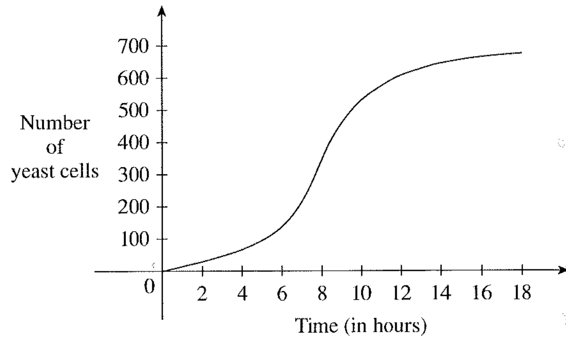
45. $f(x) = \cos x + \frac{1}{2}\cos 2x, \quad 0 \leqslant x \leqslant 2\pi$

46. $f(x) = x^3(x - 2)^4$

CAS **47–48** Estimate the intervals of concavity to one decimal place by using a computer algebra system to compute and graph f''.

47. $f(x) = \dfrac{x^4 + x^3 + 1}{\sqrt{x^2 + x + 1}}$ **48.** $f(x) = \dfrac{(x + 1)^3(x^2 + 5)}{(x^3 + 1)(x^2 + 4)}$

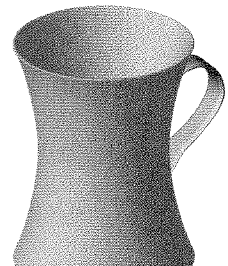
49. A graph of a population of yeast cells in a new laboratory culture as a function of time is shown.
(a) Describe how the rate of population increase varies.
(b) When is this rate highest?
(c) On what intervals is the population function concave upward or downward?
(d) Estimate the coordinates of the inflection point.



50. Let $f(t)$ be the temperature at time t where you live and suppose that at time $t = 3$ you feel uncomfortably hot. How do you feel about the given data in each case?
(a) $f'(3) = 2, \quad f''(3) = 4$
(b) $f'(3) = 2, \quad f''(3) = -4$
(c) $f'(3) = -2, \quad f''(3) = 4$
(d) $f'(3) = -2, \quad f''(3) = -4$

51. Let $K(t)$ be a measure of the knowledge you gain by studying for a test for t hours. Which do you think is larger, $K(8) - K(7)$ or $K(3) - K(2)$? Is the graph of K concave upward or concave downward? Why?

52. Coffee is being poured into the mug shown in the figure at a constant rate (measured in volume per unit time). Sketch a rough graph of the depth of the coffee in the mug as a function of time. Account for the shape of the graph in terms of concavity. What is the significance of the inflection point?



53. Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ that has a local maximum value of 3 at $x = -2$ and a local minimum value of 0 at $x = 1$.

54. Show that the curve $y = (1 + x)/(1 + x^2)$ has three points of inflection and they all lie on one straight line.

55. (a) If the function $f(x) = x^3 + ax^2 + bx$ has the local minimum value $-\frac{2}{9}\sqrt{3}$ at $x = 1/\sqrt{3}$, what are the values of a and b?
(b) Which of the tangent lines to the curve in part (a) has the smallest slope?

56. For what values of a and b is $(2, 2.5)$ an inflection point of the curve $x^2y + ax + by = 0$? What additional inflection points does the curve have?

57. Show that the inflection points of the curve $y = x \sin x$ lie on the curve $y^2(x^2 + 4) = 4x^2$.

58–60 Assume that all of the functions are twice differentiable and the second derivatives are never 0.

58. (a) If f and g are concave upward on I, show that $f + g$ is concave upward on I.
(b) If f is positive and concave upward on I, show that the function $g(x) = [f(x)]^2$ is concave upward on I.

59. (a) If f and g are positive, increasing, concave upward functions on I, show that the product function fg is concave upward on I.
(b) Show that part (a) remains true if f and g are both decreasing.
(c) Suppose f is increasing and g is decreasing. Show, by giving three examples, that fg may be concave upward, concave downward, or linear. Why doesn't the argument in parts (a) and (b) work in this case?

60. Suppose f and g are both concave upward on $(-\infty, \infty)$. Under what condition on f will the composite function $h(x) = f(g(x))$ be concave upward?

---

61. Show that $\tan x > x$ for $0 < x < \pi/2$. [*Hint:* Show that $f(x) = \tan x - x$ is increasing on $(0, \pi/2)$.]

62. Prove that, for all $x > 1$,

$$2\sqrt{x} > 3 - \frac{1}{x}$$

63. Show that a cubic function (a third-degree polynomial) always has exactly one point of inflection. If its graph has three x-intercepts x_1, x_2, and x_3, show that the x-coordinate of the inflection point is $(x_1 + x_2 + x_3)/3$.

64. For what values of c does the polynomial $P(x) = x^4 + cx^3 + x^2$ have two inflection points? One inflection point? None? Illustrate by graphing P for several values of c. How does the graph change as c decreases?

65. Prove that if $(c, f(c))$ is a point of inflection of the graph of f and f'' exists in an open interval that contains c, then $f''(c) = 0$. [*Hint:* Apply the First Derivative Test and Fermat's Theorem to the function $g = f'$.]

66. Show that if $f(x) = x^4$, then $f''(0) = 0$, but $(0, 0)$ is not an inflection point of the graph of f.

67. Show that the function $g(x) = x|x|$ has an inflection point at $(0, 0)$ but $g''(0)$ does not exist.

68. Suppose that f''' is continuous and $f'(c) = f''(c) = 0$, but $f'''(c) > 0$. Does f have a local maximum or minimum at c? Does f have a point of inflection at c?

69. Suppose f is differentiable on an interval I and $f'(x) > 0$ for all numbers x in I except for a single number c. Prove that f is increasing on the entire interval I.

70. For what values of c is the function

$$f(x) = cx + \frac{1}{x^2 + 3}$$

increasing on $(-\infty, \infty)$?

71. The three cases in the First Derivative Test cover the situations one commonly encounters but do not exhaust all possibilities. Consider the functions f, g, and h whose values at 0 are all 0 and, for $x \neq 0$,

$$f(x) = x^4 \sin\frac{1}{x} \qquad g(x) = x^4\left(2 + \sin\frac{1}{x}\right)$$

$$h(x) = x^4\left(-2 + \sin\frac{1}{x}\right)$$

(a) Show that 0 is a critical number of all three functions but their derivatives change sign infinitely often on both sides of 0.
(b) Show that f has neither a local maximum nor a local minimum at 0, g has a local minimum, and h has a local maximum.

3.4 Limits at Infinity; Horizontal Asymptotes

In Sections 1.5 and 1.7 we investigated infinite limits and vertical asymptotes. There we let x approach a number and the result was that the values of y became arbitrarily large (positive or negative). In this section we let x become arbitrarily large (positive or negative) and see what happens to y. We will find it very useful to consider this so-called *end behavior* when sketching graphs.