METU - NCC

LINEAR ALGEBRA MIDTERM 1	
Code : *MAT 260* Acad.Year: *2012-2013* Semester : *Spring* Date : *20.03.2013* Time : *17:40* Duration : *110 min*	Last Name: Name : Student No.: Department: Section: Signature : 5 QUESTIONS ON 5 PAGES TOTAL 100 POINTS

1. (22)	2. (20)	3. (22)	4. (24)	5. (12)		

1. *(12+10pts)* (a) Let A, B, C be three sets.
Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

($\subseteq$) If $A \cap (B \cup C) = \emptyset$, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
Otherwise pick $\underline{x} \in A \cap (B \cup C)$. $\underline{x} \in A$ and $\underline{x} \in (B \cup C)$.
<u>Case 1</u>: $\underline{x} \in B$
Then $\underline{x} \in A \cap B$ so $\underline{x} \in (A \cap B) \cup (A \cap C)$.
<u>Case 2</u>: $\underline{x} \in C$
Then $\underline{x} \in A \cap C$ so $\underline{x} \in (A \cap B) \cup (A \cap C)$.
($\supseteq$) If $(A \cap B) \cup (A \cap C) = \emptyset$ then $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$
Otherwise pick $\underline{x} \in (A \cap B) \cup (A \cap C)$.
<u>Case 1</u>: $\underline{x} \in (A \cap B)$
Then $\underline{x} \in A$ and $\underline{x} \in B$. So $\underline{x} \in B \cup C$. Thus $\underline{x} \in A \cap (B \cup C)$.
<u>Case 2</u>: $\underline{x} \in (A \cap C)$
Then $\underline{x} \in A$ and $\underline{x} \in C$. So $\underline{x} \in B \cup C$. Thus $\underline{x} \in A \cap (B \cup C)$. ∎

(b) Suppose that $\mathcal{V}$ is a vector space, $\mathbf{v} \in \mathcal{V}$, and $a, b \in \mathbb{R}$ with $a \neq b$.
Show that if $a \cdot \mathbf{v} = b \cdot \mathbf{v}$ then $\mathbf{v} = \mathbf{0}$.

Suppose $a\underline{v} = b\underline{v}$. Then $a\underline{v} - b\underline{v} = b\underline{v} - b\underline{v}$
$(a-b)\cdot\underline{v} = \underline{0}$ by Axioms 4, 6

We proved in class that this means either $(a-b) = 0$ or $\underline{v} = \underline{0}$.
Since $a \neq b$ we must have $\underline{v} = \underline{0}$. ∎

(Alternately... since $a \neq b$, $a-b \neq 0$. Thus we have
$$\tfrac{1}{a-b}\cdot(a-b)\cdot\underline{v} = \tfrac{1}{a-b}\cdot\underline{0}$$
$$1\cdot\underline{v} = \underline{0}$$
$$\underline{v} = \underline{0}.$$ ∎)

2. *(10+10pts)* (a) Recall that $\mathrm{Fun}(\mathbb{R}, \{4\})$ is the set of real-valued functions on $\mathbb{R}$ with $f(4) = 0$.
Is $\mathrm{Fun}(\mathbb{R}, \{4\})$ a subspace of $\mathrm{Fun}(\mathbb{R})$? (Prove your answer.)

$\mathrm{Fun}(\mathbb{R}, \{4\})$ is a subspace of $\mathrm{Fun}(\mathbb{R})$.

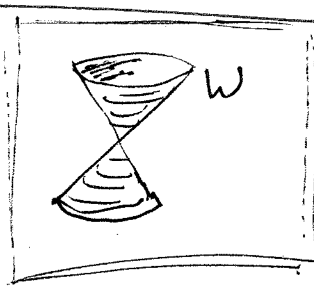
- The zero function has $\underline{0}(4) = 0$ so $\underline{0} \in \mathrm{Fun}(\mathbb{R}, \{4\})$
  Thus $\mathrm{Fun}(\mathbb{R}, \{4\}) \neq \emptyset$

- If $\underline{f}, \underline{g} \in \mathrm{Fun}(\mathbb{R}, \{4\})$ then $\underline{f}(4) = \underline{g}(4) = 0$.
  So $$(\underline{f+g})(4) = \underline{f}(4) + \underline{g}(4) = 0 + 0 = 0$$
  Thus $\underline{f+g} \in \mathrm{Fun}(\mathbb{R}, \{4\})$, so it is closed under $+$.

- If $\underline{f} \in \mathrm{Fun}(\mathbb{R}, \{4\})$ then $\underline{f}(4) = 0$
  So $$(c\underline{f})(4) = c\,\underline{f}(4) = c \cdot 0 = 0$$
  Thus $c\underline{f} \in \mathrm{Fun}(\mathbb{R}, \{4\})$, so it is closed under scaling.

$\blacksquare$

(b) Let $W = \{(x, y, z) \text{ with } z^2 = x^2 + y^2\} \subset \mathbb{R}^3$.
Is W a subspace of $\mathbb{R}^3$? (Prove your answer.)



W is not a subspace of $\mathbb{R}^3$.
(In particular, it is not closed under addition.)

Example: $(1, 0, 1)$ and $(0, 1, 1) \in W$
because $1^2 = 1^2 + 0^2$ and $1^2 = 0^2 + 1^2$.

However, $(1, 0, 1) + (0, 1, 1) = (1, 1, 2) \notin W$
because $2^2 \neq 1^2 + 1^2$.

$\blacksquare$

Name: ID:

3. *(10+12pts)* (a) Let E and F be finite subsets of a vector space $\mathcal{V}$. Show that if $E \subseteq \text{Span}(F)$, then $\text{Span}(E) \subseteq \text{Span}(F)$.

Suppose that $E \subseteq \text{Span}(F)$. If $\underline{v} \in \text{Span}(E)$ then

$$\underline{v} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + \cdots + a_n \underline{e}_n \quad \text{where } \underline{e}_1, \ldots, \underline{e}_n \in E.$$

However, each $\underline{e}_i \in \text{Span}(F)$ so

$$\underline{e}_i = b_{i1} \underline{f}_1 + b_{i2} \underline{f}_2 + \cdots + b_{ik} \underline{f}_k \quad \text{where } F = \{\underline{f}_1, \ldots, \underline{f}_k\}.$$

Thus

$$\begin{aligned}\underline{v} &= a_1 (b_{11} \underline{f}_1 + b_{12} \underline{f}_2 + \cdots + b_{1k} \underline{f}_k) \\ &\quad + a_2 (b_{21} \underline{f}_1 + b_{22} \underline{f}_2 + \cdots + b_{2k} \underline{f}_k) \\ &\quad + \cdots \\ &\quad + a_n (b_{n1} \underline{f}_1 + b_{n2} \underline{f}_2 + \cdots + b_{nk} \underline{f}_k) \\ &= (a_1 b_{11} + a_2 b_{21} + \cdots + a_n b_{n1}) \underline{f}_1 + \cdots + (a_1 b_{1k} + \cdots + a_n b_{nk}) \underline{f}_k.\end{aligned}$$

$\blacksquare$

(b) Let $\mathcal{V}$ be a vector space and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$ with $\mathbf{u} + 2\mathbf{v} + 3\mathbf{w} = \mathbf{0}$. Show that $\text{Span}(\{\mathbf{u}, \mathbf{v}\}) = \text{Span}(\{\mathbf{v}, \mathbf{w}\})$.

($\subseteq$) $\underline{v} \in \text{Span}(\{\underline{v}, \underline{w}\})$

Also $\underline{u} = -2\underline{v} - 3\underline{w} \in \text{Span}(\{\underline{v}, \underline{w}\})$

Since $\{\underline{u}, \underline{v}\} \subset \text{Span}(\{\underline{v}, \underline{w}\})$, $\text{Span}(\{\underline{u}, \underline{v}\}) \subseteq \text{Span}(\{\underline{v}, \underline{w}\})$.

($\supseteq$) $\underline{v} \in \text{Span}(\{\underline{u}, \underline{v}\})$

Also $\underline{w} = -\frac{1}{3} \cdot \underline{u} - \frac{2}{3} \cdot \underline{v} \in \text{Span}(\{\underline{u}, \underline{v}\})$

Since $\{\underline{v}, \underline{w}\} \subset \text{Span}(\{\underline{u}, \underline{v}\})$, $\text{Span}(\{\underline{v}, \underline{w}\}) \subseteq \text{Span}(\{\underline{u}, \underline{v}\})$.

$\blacksquare$

4. *(14+10pts)* (a) Show that $\{1+x+x^2,\ 1-x+x^2,\ 1-x^2\}$ is a basis for $\mathcal{P}_2(\mathbb{R})$.

- First we will show that Span $\{1+x+x^2,\ 1-x+x^2,\ 1-x^2\} = P_2(\mathbb{R})$:

$$a_0 + a_1 x + a_2 x^2 = c_1(1+x+x^2) + c_2(1-x+x^2) + c_3(1-x^2)$$
$$= (c_1+c_2+c_3) + (c_1-c_2)x + (c_1+c_2-c_3)x^2$$

$$a_0 = c_1+c_2+c_3 \qquad a_0 - a_2 = 2c_3 \rightsquigarrow \boxed{c_3 = \tfrac{1}{2}(a_0-a_2)}$$
$$a_1 = c_1 - c_2 \qquad a_0 + a_2 = 2c_1 + 2c_2$$
$$a_2 = c_1+c_2-c_3 \qquad (a_1 = c_1 - c_2)2$$
$$a_0 + 2a_1 + a_2 = 4c_1 \rightsquigarrow \boxed{c_1 = \tfrac{1}{4}(a_0+2a_1+a_2)}$$
$$\rightsquigarrow \boxed{c_2 = \tfrac{1}{4}(a_0-2a_1+a_2)}$$

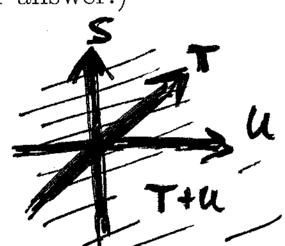
Since $a_0 + a_1 x + a_2 x^2 = \tfrac{1}{4}(a_0+2a_1+a_2)\cdot(1+x+x^2)$
$+ \tfrac{1}{4}(a_0-2a_1+a_2)\cdot(1-x+x^2)$
$+ \tfrac{1}{2}(a_0-a_2)\cdot(1-x^2)$

these polynomials span P_2.

- Now we show independence: From our work above, if $a_0 = a_1 = a_2 = 0$ then $c_1 = c_2 = c_3 = 0$. So these polynomials are independent. ▨

(b) Let $\mathcal{S}, \mathcal{T}, \mathcal{U}$ be subspaces of a vector space $\mathcal{V}$.
Is it always true that $\mathcal{S} \cap (\mathcal{T}+\mathcal{U}) = (\mathcal{S}\cap\mathcal{T}) + (\mathcal{S}\cap\mathcal{U})$? (Prove your answer.)

This is FALSE.



Consider three lines in $\mathbb{R}^2$:
$T + U = \mathbb{R}^2$ but $S \cap T = \{0\}$
and $S \cap U = \{0\}$

Explicitly, let $S = \text{Span}\{(1,1)\}$
$T = \text{Span}\{(1,0)\}$
$U = \text{Span}\{(0,1)\}$

Then $T+U = \text{Span}\{(1,0),(0,1)\} = \mathbb{R}^2$
but $S \cap T = \{(0,0)\}$
and $S \cap U = \{(0,0)\}$ so $(S\cap T)+(S\cap U) = \{(0,0)\}$ ▨

5. *(12pts)* Let $S = \{1, 2, 3, 4\}$ and $\mathcal{V} = \mathrm{Fun}(S)$.
Let $\chi_t(s)$ denote the characteristic function whose value is 1 if $s = t$ and 0 if $s \neq t$.
Let $E = \{\chi_1(t) - \chi_2(t),\ \chi_2(t) - \chi_3(t),\ \chi_3(t) - \chi_4(t),\ \chi_4(t) - \chi_1(t)\}$.
Is E linearly independent? If not, find a linearly independent subset with the same span.

E is <u>not</u> linearly independent.

$$0 = a\left(\chi_1(t) - \chi_2(t)\right) + b\left(\chi_2(t) - \chi_3(t)\right) + c\left(\chi_3(t) - \chi_4(t)\right) + d\left(\chi_4(t) - \chi_1(t)\right)$$

$$= (a-d)\chi_1(t) + (b-a)\chi_2(t) + (c-b)\chi_3(t) + (d-c)\chi_4(t)$$

(*) Evaluating at $t=1$ gives: $0 = a - d$

$$\left.\begin{array}{ll} t=1 \text{ gives: } & 0 = a-d \\ t=2 \text{ gives: } & 0 = b-a \\ t=3 \text{ gives: } & 0 = c-b \\ t=4 \text{ gives: } & 0 = d-c \end{array}\right\} \quad a=b=c=d.$$

For example,

$$0 = 1\cdot(\chi_1 - \chi_2) + 1\cdot(\chi_2 - \chi_3) + 1\cdot(\chi_3 - \chi_4) + 1\cdot(\chi_4 - \chi_1)$$

We can rearrange the above dependence relation to get:

$$(\chi_4 - \chi_1) = -(\chi_1 - \chi_2) - (\chi_2 - \chi_3) - (\chi_3 - \chi_4)$$

So removing $\chi_4 - \chi_1$ from E does not change the span.

This has the effect of setting $d=0$ in (*) above.
Thus $a=b=c=0$. So the remaining set

$$E' = \{\chi_1 - \chi_2,\ \chi_2 - \chi_3,\ \chi_3 - \chi_4\}$$

is linearly independent.