M E T U
Northern Cyprus Campus

<table>
<tr><td colspan="3">Math 219   Differential Equations   Final Exam   04.06.2015</td></tr>
<tr><td>Last Name :
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Q1 (25 pts) Find the solution to the IVP $\begin{cases} y'' + y = u_{\pi/2}(t) + 2\delta(t - 3\pi/2) - u_{2\pi}(t), \\ y(0) = y'(0) = 0. \end{cases}$

$$(s^2+1)Y(s) = \frac{e^{-\frac{\pi}{2}s}}{s} + 2e^{-\frac{3\pi}{2}s} - \frac{e^{-2\pi s}}{s}$$

Since $\dfrac{1}{s(s^2+1)} = \dfrac{1}{s} - \dfrac{s}{s^2+1}$, it follows that

$$Y(s) = \frac{e^{-\frac{\pi}{2}s}}{s} - \frac{s\,e^{-\frac{\pi}{2}s}}{s^2+1} + \frac{2e^{-\frac{3\pi}{2}s}}{s^2+1} - \frac{e^{-2\pi s}}{s} + \frac{s\,e^{-2\pi s}}{s^2+1}.$$

Based on the Shifting Formula, we derive that

$$y(t) = u_{\frac{\pi}{2}}(t)\left(1 - \cos\left(t - \tfrac{\pi}{2}\right)\right) + 2u_{\frac{3\pi}{2}}(t)\sin\left(t - \tfrac{3\pi}{2}\right)$$
$$- u_{2\pi}(t)\left(1 - \cos(t - 2\pi)\right)$$

Q2 **(25 pts)** The problem has unrelated parts.

a) Find $F(s) = \mathfrak{L}\{f(t)\}$ for the function $f(t) = \begin{cases} t-3, & 0 \le t \le 3; \\ t^2+1, & 3 < t < 5; \\ 1, & t \ge 5. \end{cases}$

$f(t) = (t-3) + u_3(t)(t^2 - t + 4) - u_5(t)\,t^2 =$

$= t - 3 + u_3(t)\left((t-3)^2 + 5(t-3) + 10\right) - u_5(t)\left((t-5)^2 + 10(t-5) + 25\right)$

$F(s) = \frac{1}{s^2} - \frac{3}{s} + e^{-3s}\mathcal{L}\{t^2 + 5t + 10\} - e^{-5s}\mathcal{L}\{t^2 + 10t + 25\} =$

$= \frac{1}{s^2} - \frac{3}{s} + e^{-3s}\left(\frac{2}{s^3} + \frac{5}{s^2} + \frac{10}{s}\right) - e^{-5s}\left(\frac{2}{s^3} + \frac{10}{s^2} + \frac{25}{s}\right).$

b) Find $f(t) = \mathfrak{L}^{-1}\{F(s)\}$ for the function $F(s) = \frac{e^{-3s}(s+21)}{s^2+2s+5}$.

Note that $f(t) = u_3(t)\,g(t-3)$, where $G(s) = \frac{s+21}{s^2+2s+5}$

$= \frac{s+1}{(s+1)^2+4} + 10\,\frac{2}{(s+1)^2+4}$. Therefore

$g(t) = e^{-t}\cos(2t) + 10e^{-t}\sin(2t)$. Thus

$f(t) = u_3(t)\left(e^{-(t-3)}\cos(2(t-3)) + 10e^{-(t-3)}\sin(2(t-3))\right)$

c) Compute the convolution $u_2(t) * \sin(t)$. Put $f(t) = u_2(t) * \sin(t)$.

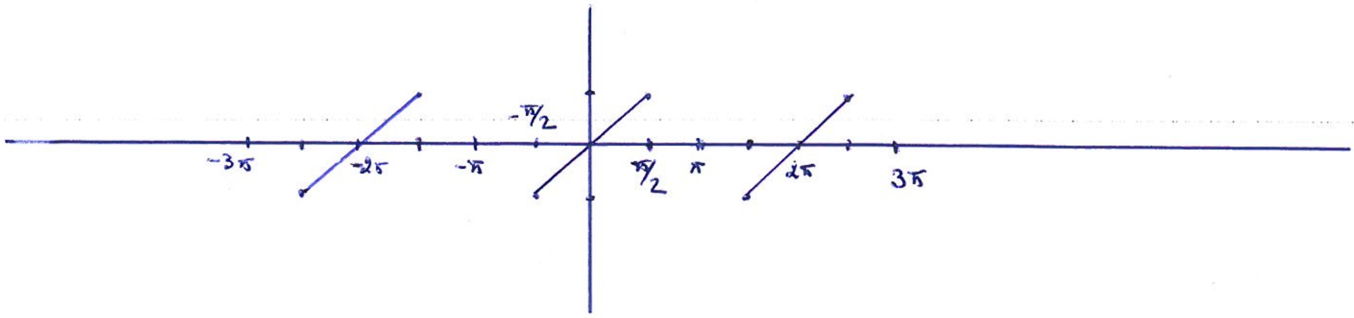
Then $F(s) = \mathcal{L}\{u_2(t)\}\mathcal{L}\{\sin(t)\} = \frac{e^{-2s}}{s}\,\frac{1}{s^2+1} =$

$= \frac{e^{-2s}}{s} - \frac{s}{s^2+1}e^{-2s}$. Apply $\mathcal{L}^{-1}$:

$f(t) = u_2(t) - u_2(t)\cos(t-2)$.

Q3 (25 pts) Consider the function $f(x) = \begin{cases} x, & 0 < x < \pi/2; \\ 0, & \pi/2 < x < \pi. \end{cases}$

a) Graph the odd periodic extension of $f(x)$ on the interval $[-3\pi, 3\pi]$.



b) Find the Fourier series $S_f(x)$ of the extended function $f(x)$ from a).

Since we have an odd extension, $S_f(x)$ is the sine series of the original function f, that is, $S_f(x) = \sum_{m=1}^{\infty} b_m \sin(mx)$ with

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi/2} x\sin(nx)\,dx =$$

$$= \frac{-2}{\pi n}\int_0^{\pi/2} x\cos'(nx)\,dx = \frac{-2}{\pi n}\, x\cos(nx)\Big|_0^{\pi/2} + \frac{2}{\pi n}\int_0^{\pi/2}\cos(nx)\,dx$$

$$= \frac{-1}{n}\cos\left(\frac{\pi n}{2}\right) + \frac{2}{\pi n^2}\sin\left(\frac{\pi n}{2}\right) =$$

$$= \begin{cases} \dfrac{(-1)^{n/2+1}}{n} & \text{if } n \text{ is even} \\[2ex] \dfrac{2(-1)^{\frac{n-1}{2}}}{\pi n^2} & \text{if } n \text{ is odd.} \end{cases}$$

c) Compute $S_f(5\pi/2)$ and $S_f(-16)$. Based on Fourier's Theorem, we have $S_f(5\pi/2) = S_f\left(\frac{\pi}{2} + 2\pi\right) = \frac{1}{2}\left(f\left(\frac{\pi}{2}+\right) + f\left(\frac{\pi}{2}-\right)\right)$

$$= \frac{1}{2}\left(0 + \frac{\pi}{2}\right) = \pi/4;$$

$$S_f(-16) = S_f(-16 + 6\pi) = f(-16 + 6\pi) = 0.$$

Q4 (25 pts) Find the general solution to the following linear system

$$\mathbf{x}' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix}, \; t > 0.$$

Put $A = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$, $\vec{b}(t) = \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix}$.

$$\Delta(\lambda) = \begin{vmatrix} 4-\lambda & -2 \\ 8 & -4-\lambda \end{vmatrix} = -(4-\lambda)(4+\lambda)+16 = -(16-\lambda^2)+16 = \lambda^2,$$

$\sigma(A) = \{0^{(2)}\}$, $V_{0,1} = \ker(A) = \{2x = y\} \underset{\neq}{\subset} V_{0,2} = \mathbb{R}^2$.

Take $f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin V_{0,1}$. Then $f_2 = A f_1 = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$. So

$P = \begin{bmatrix} 1 & 4 \\ 0 & 8 \end{bmatrix}$, $J = P^{-1} A P = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and

$$\Psi(t) = P e^{Jt} = \begin{bmatrix} 1 & 4 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} 1+4t & 4 \\ 8t & 8 \end{bmatrix},$$

$W(t) = 8 + 32t - 32t = 8$. For a special solution we set $\vec{Y}(t) = \Psi(t)\vec{C}(t)$ with

$$\Psi(t)\vec{C}'(t) = \vec{b}(t) \Rightarrow \vec{C}'(t) = \Psi(t)^{-1}\vec{b}(t) =$$

$$= \frac{1}{8}\begin{bmatrix} 8 & -4 \\ -8t & 1+4t \end{bmatrix} \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix} = \begin{bmatrix} t^{-3} + \frac{1}{2} t^{-2} \\ -\frac{9}{8} t^{-2} - \frac{1}{2} t^{-1} \end{bmatrix} \Rightarrow$$

$$\vec{C}(t) = \begin{bmatrix} -\frac{1}{2}(t^{-1} + t^{-2}) \\ \frac{9}{8} t^{-1} - \frac{1}{2}\ln(t) \end{bmatrix}$$

Whence

$$\vec{x}(t) = \begin{bmatrix} 1+4t & 4 \\ 8t & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \vec{Y}(t)$$

is the general solution.