M E T U
Northern Cyprus Campus

Math 219	Differential Equations	Midterm Exam II	03.05.2014
Last Name: Name : KEY Student No:	Dept./Sec. : Time : 15: 40 Duration : 100 *minutes*	Signature	
3 QUESTIONS		TOTAL 100 POINTS	

1	2	3	4	5			

Q1 (20=10+4+2+5 pts.) Find the fundamental matrices $\Psi(t)$, $\Phi(t)$, and e^A for the following 2×2-system $\mathbf{x}'(t) = A\mathbf{x}(t)$ with the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Sketch the phase portrait.

$$A-\lambda = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}, \quad \Delta(\lambda) = (\lambda-1)(\lambda-3), \quad \sigma(A) = \{1^{①}, 3^{①}\}$$

$$\lambda = 1 \Rightarrow A-1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad V_1 = \ker(A-1) = \{x+y=0\}, \quad \vec{f}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

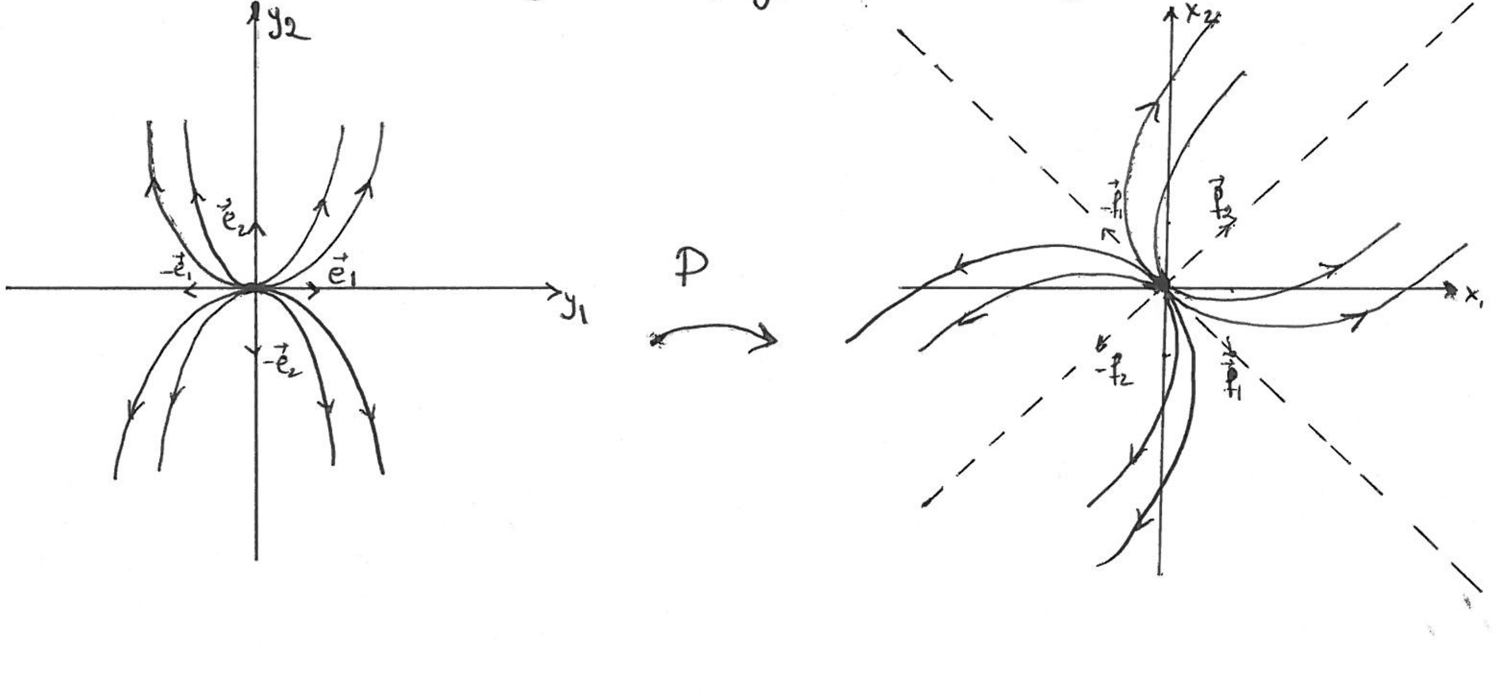
$$\lambda = 3 \Rightarrow A-3 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad V_3 = \ker(A-3) = \{x=y\}, \quad \vec{f}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $\Psi(t) = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix}$

and $\Phi(t) = \Psi(t)P^{-1} = \frac{1}{2}\begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{e^t+e^{3t}}{2} & \frac{-e^t+e^{3t}}{2} \\ \frac{-e^t+e^{3t}}{2} & \frac{e^t+e^{3t}}{2} \end{bmatrix}$,

$$\Phi(0) = I_2 \Rightarrow e^A = \Phi(1) = \begin{bmatrix} \frac{e+e^3}{2} & \frac{-e+e^3}{2} \\ \frac{-e+e^3}{2} & \frac{e+e^3}{2} \end{bmatrix}$$

Phase Portrait: $y_1 = C_1e^t$, $y_2 = C_2 e^{3t} \Rightarrow y_2 = C y_1^3$



Q2 (20=15+5 pts.) Find the fundamental matrix $\Psi(t)$ and sketch the phase portrait for the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $A = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}$.

$$A-\lambda = \begin{bmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{bmatrix}, \quad \Delta(\lambda) = \lambda^2+2\lambda+2 \Rightarrow \sigma(A) = \{-1 \pm i\},$$

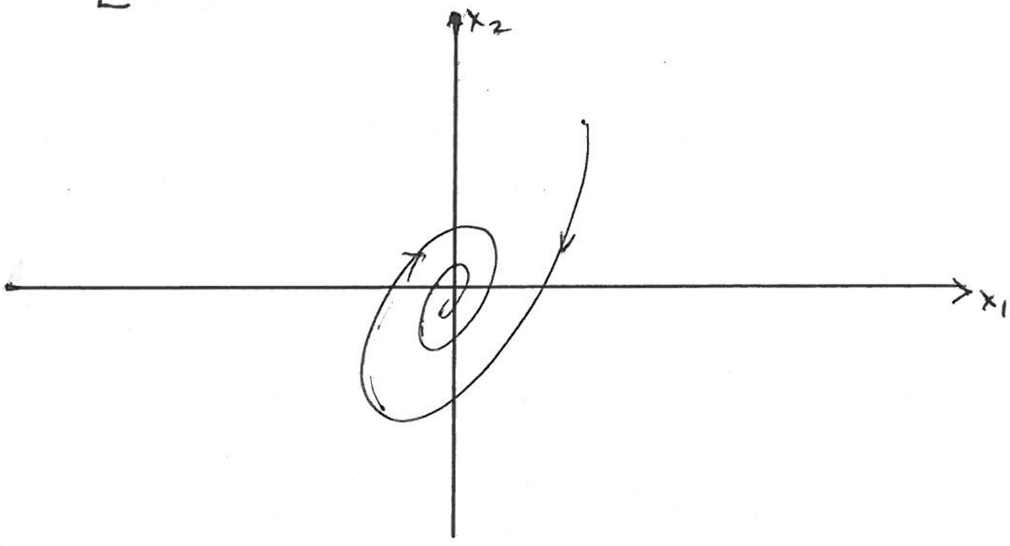
$$A-(-1+i) = \begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix} \sim \begin{bmatrix} 2-i & -1 \\ 0 & 0 \end{bmatrix}, \quad V_{-1+i} = \{(2-i)x=y\}, \quad \vec{f} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$

So we have the complex solution $\vec{x}(t) = \vec{f} e^{-t} e^{it} \Rightarrow$

$$\vec{x}(t) = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)(\cos(t) + i\sin(t)) e^{-t} =$$

$$= \left(\begin{bmatrix} \cos(t) \\ 2\cos(t)+\sin(t) \end{bmatrix} + i \begin{bmatrix} \sin(t) \\ 2\sin(t)-\cos(t) \end{bmatrix}\right) e^{-t},$$

$$\Psi(t) = \begin{bmatrix} e^{-t}\cos(t) & e^{-t}\sin(t) \\ e^{-t}(2\cos(t)+\sin(t)) & e^{-t}(2\sin(t)-\cos(t)) \end{bmatrix}$$



Q3 (10 pts.) Find e^A with $A = \begin{bmatrix} 0 & 2 & -3 \\ 0 & 0 & 9 \\ 0 & 0 & 0 \end{bmatrix}$ using the power series definition of the matrix exponent. Note that $A^3 = 0$. Therefore

$$e^A = I + A + \frac{1}{2}A^2 = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix}$$

Q4 (25 pts.) Find the general solution to $\mathbf{x}'(t) = \begin{bmatrix} -1 & -1 & -1 \\ -2 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \mathbf{x}(t)$.

$$A-\lambda = \begin{bmatrix} -1-\lambda & -1 & -1 \\ -2 & -1-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix}, \quad \Delta(\lambda) = -(\lambda+1)^3 + 2 + \lambda+1 + 2(\lambda+1)$$

$$= -(\lambda+1)^3 + 3\lambda + 5 =$$

$$= -\lambda^3 - 3\lambda^2 - 3\lambda - 1 + 3\lambda + 5 = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda-1)(\lambda+2)^2,$$

$$\sigma(A) = \{-2^{(2)}, 1^{(1)}\}.$$

$$\lambda = -2, \quad A+2 = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$V_{-2,1} = \ker(A+2) = \{x=0, y=-z\}$, $m(-2) = 1 < alg(-2) = 2$.

But $(A+2)^2 = \begin{bmatrix} 3 & -3 & -3 \\ -4 & 4 & 4 \\ -2 & 2 & 2 \end{bmatrix}$ and $V_{-2,2} = \ker(A+2)^2 = \{x=y+z\}$

We got a chain $V_{-2,1} \lneqq V_{-2,2}$. Take

$$\{x=0, y=-z\} \lneqq \{x=y+z\}.$$

$\vec{f}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in V_{-2,2} - V_{-2,1}$. Then $\vec{f}_2 = (A+2)\vec{f}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \in V_{-2,1}$.

$$\lambda = 1, \quad A-1 = \begin{bmatrix} -2 & -1 & -1 \\ -2 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$V_1 = \ker(A-1) = \{2x + 3z = 0, y = 2z\}$, $\vec{f}_3 = \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}$

$$\Psi(t) = Pe^{Jt} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & -1 & 4 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 & 0 \\ te^{-2t} & e^{-2t} & 0 \\ 0 & 0 & e^t \end{bmatrix} =$$

$$= \begin{bmatrix} e^{-2t} & 0 & -3e^t \\ -te^{-2t} & -e^{-2t} & 4e^t \\ (1+t)e^{-2t} & e^{-2t} & 2e^t \end{bmatrix}, \quad \vec{x}(t) = \Psi(t)\vec{c}$$

Q5 (25=20+5 pts.) Find a general solution to $\mathbf{x}'(t) = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1/t^3 \\ -1/t^2 \end{bmatrix}$, $t > 0$.

Sketch the phase portrait to the related homogeneous system.

$A - \lambda = \begin{bmatrix} 2-\lambda & -1 \\ 4 & -2-\lambda \end{bmatrix}$, $\Delta(\lambda) = \lambda^2$, $\sigma(A) = \{0^{(2)}\}$

$\lambda = 0 \Rightarrow A = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$, $V_{0,1} = \ker(A) = \{2x = y\}$,

$V_{0,2} = \ker(A^2) = \mathbb{C}^2$, $\vec{f}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{f}_2 = A\vec{f}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$,

$J = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow \Psi(t) = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} 1+2t & 2 \\ 4t & 4 \end{bmatrix}$

Put $\vec{y}(t) = \Psi(t)\vec{c}(t)$. Then $\Psi(t)\vec{c}\,'(t) = \vec{b}(t)$ and

$c_1'(t) = \frac{1}{4} \begin{vmatrix} t^{-3} & 2 \\ -t^{-2} & 4 \end{vmatrix} = \frac{1}{4}(4t^{-3} + 2t^{-2}) = t^{-3} + \frac{1}{2}t^{-2}$,

$c_1(t) = -\frac{1}{2}t^{-2} - \frac{1}{2}t^{-1} + c_1$ $\left(W(t) = \det \Psi(t) = 4 \right)$

$c_2'(t) = \frac{1}{4} \begin{vmatrix} 1+2t & t^{-3} \\ 4t & -t^{-2} \end{vmatrix} = \frac{1}{4}\left(-\frac{1+2t}{t^2} - \frac{4}{t^2} \right) = -\frac{5+2t}{4t^2}$

$c_2(t) = -\frac{5}{4}\frac{1}{t} - \frac{1}{2}\ln(t) + C_2$

So, $\vec{y}(t) = \begin{bmatrix} 1+2t & 2 \\ 4t & 4 \end{bmatrix} \begin{bmatrix} -\frac{t^{-1}+t^{-2}}{2} \\ -\frac{5}{4}t^{-1} - \frac{1}{2}\ln(t) \end{bmatrix}$ and

$\vec{x}(t) = \Psi(t)\vec{C} + \vec{y}(t)$ is the general solution.

$y_1 = C_1$, $y_2 = C_1 t + C_2$