METU - NCC

	Differential Equations Midterm I	
Code : *Math 219* Acad.Year: *2011-2012* Semester : *Fall* Date : *10.30.2011* Time : *13:40* Duration : *120 minutes*	Last Name: Name : Department: Signature:	Student No.: Section:
	8 QUESTIONS ON 6 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6	7	8		

1. (5pts) The differential equation $xy'' - 3xy' - 3y = 0$ has a solution of the form $y = xe^{ax}$. Find a.

$y = x e^{ax}$

$y' = 1.e^{ax} + xae^{ax} = (1+ax)e^{ax}$

$y'' = ae^{ax} + ae^{ax} + a^2xe^{ax} = (2a + a^2x)e^{ax}$

$x(2a + a^2x)e^{ax} - 3x(1+ax)e^{ax} - 3xe^{ax} = 0$

$(a-3)(2+ax)xe^{ax} = 0 \Rightarrow a - 3 = 0$

$a = 3$

2. (5+5pts) **a.** Give the general solution to the differential equation $\frac{dy}{dt} = y^{2/5}$

$\frac{dy}{dt} = y^{2/5} \Rightarrow y^{-2/5}dy = dt$

$\frac{y^{3/5}}{3/5} = t + C \Rightarrow y = \left[\frac{3}{5}(t+C)\right]^{5/3}$

b. Give an initial value $y(x_0) = y_0$ where the solution from part **a** is not the unique solution. Explain why.

If $\frac{dy}{dt} = f(y)$, then to have a unique solution both $f(y)$ and $f'(y)$ must be continuous. $y^{2/5}$ is continous everywhere but $(y^{2/5})' = y^{-3/5}$ is continuo[us] everywhere except $y = 0$

So, for any given initial value as $y(x_0) = 0$, solution

3. (5+5+5pts) Give the general solution for the following differential equations.

a. $\frac{dy}{dx} = \frac{3x^2 - 2xy}{y + x^2 + 3}$ $\Rightarrow \underbrace{(3x^2 - 2xy)}_{M}dx - \underbrace{(y + x^2 + 3)}_{N}dy = 0$

$\frac{\partial M}{\partial y} = -2x = \frac{\partial N}{\partial x}$ So, eqn is exact.

Which means there exist f such that

$\frac{\partial f}{\partial x} = 3x^2 - 2xy \Rightarrow f = x^3 - x^2y + h(y)$

$\frac{\partial f}{\partial y} = -x^2 + h'(y) = -(y + x^2 + 3) \Rightarrow h'(y) = -y - 3 \Rightarrow h(y) = -\frac{y^2}{2} - 3y$

solution is $x^3 - x^2y - \frac{y^2}{2} - 3y = C$

b. $\frac{dy}{dx} = 3\cos^2 x - \tan(x)y$

$y' + \tan(x)y = 3\cos^2 x \Rightarrow \mu = e^{\int \tan x \, dx} = e^{-\ln|\cos x|} = \frac{1}{\cos x}$

$\Rightarrow y = \frac{\int 3\cos^2 x \cdot \frac{1}{\cos x} \, dx}{\frac{1}{\cos x}} = \frac{3\sin x + C}{\frac{1}{\cos x}}$

$y = 3\sin x \cos x + C\cos x$

c. $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$ make substution $y = vx$

$\Rightarrow v'x + v = \frac{x^2 + v^2x^2}{2x\,vx} \Rightarrow v'x + v = \frac{1 + v^2}{2v}$

$v'x = \frac{1 + v^2}{2v} - v$

$x\frac{dv}{dx} = \frac{1 - v^2}{2v}$

$\Rightarrow \frac{2v}{1 - v^2}dv = \frac{dx}{x} \Rightarrow -\ln|1 - v^2| = \ln x + C_0$

$\Rightarrow x(1 - v^2) = e^{-C_0}$

$x(1 - v^2) = C$

$x\left(1 - \frac{y^2}{x^2}\right) = C$

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4. (4+10+2pts) Water polluted with 0.5 $\frac{\text{mg}}{\text{L}}$ of silver is dumped into a storage tank at a rate of 5 $\frac{\text{L}}{\text{m}}$ and is absorbed into the ground at a rate of 1 $\frac{\text{L}}{\text{m}}$. Suppose that the tank begins with 16 L of clean water, and the polluted water stops being added after 60 minutes.

a. Give a differential equation (with initial value) for $A(t)$, the amount of silver in the tank at time t.

$\frac{dA}{dt} = \frac{5}{2} - \frac{A}{16+4t}$ and $A(0) = 0$

b. Solve the differential equation from (a) to get a formula for $A(t)$ until the time when the ~~pond~~ tank is full. (at $t=60$)

$\frac{dA}{dt} + \frac{A}{16+4t} = \frac{5}{2} \Rightarrow \mu = e^{\int \frac{1}{16+4t} dt} = e^{\frac{1}{4}\ln|4+t|} = (4+t)^{1/4}$

$A = \frac{\int (4+t)^{1/4} \cdot \frac{5}{2} dt}{(4+t)^{1/4}} = \frac{\frac{5}{2} \cdot \frac{(4+t)^{5/4}}{5/4} + C}{(4+t)^{1/4}}$

$A(t) = 2(4+t) + C(4+t)^{-1/4}$

$A(0) = 0 \Rightarrow 8 + \frac{C}{\sqrt{2}} = 0 \Rightarrow C = -8\sqrt{2}$

$A(t) = 2(4+t) - 8\sqrt{2}(4+t)^{-1/4}$

c. Having too much silver in your body causes a condition called *argyria*, where silver enters your skin and hair, and your body permanently turns blue. An average weight person drinking 3 L of water a day will turn blue if the silver concentration is above 0.33 $\frac{\text{mg}}{\text{L}}$. If you regularly drink the polluted water from the tank after it is full, will you turn blue?

$A(60) = 128 - 8\sqrt{2}(64)^{-1/4} = 128 - 4 = 124$

the amount of water is : $16 + 60 \cdot 4 = 256$

So, the concentration is $\frac{124}{256} > 0.33$.

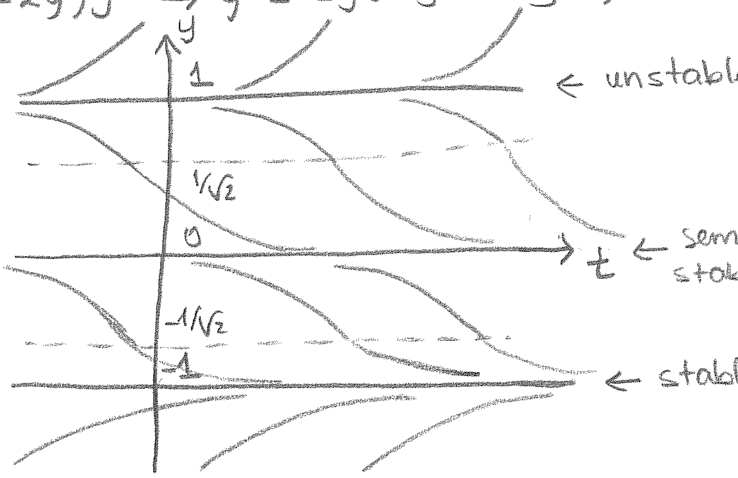
It means Yes

5. (10+3pts) Consider the differential equation $y' = y^2(y^2-1)$

a. Draw the direction field for this differential equation. Mark the equilibrium solutions and label them as (asymptotically) stable, unstable, or semi-stable.

$y' = y^2(y^2-1) \Rightarrow y'' = (4y^3-2y)y' \Rightarrow y'' = 2y^3(2y^2-1)(y^2-1)$

y		-1		$-1/\sqrt{2}$		0		$1/\sqrt{2}$		1	
y'	$+$		$+$		$-$		$-$		$+$		$+$
y''	$-$		$+$		$-$		$+$		$-$		$+$



b. If $y(0) = \frac{1}{2}$ then what is $\lim_{x\to\infty} y$?

$\lim_{x\to\infty} y = 0$

6. (7pts) Solve the following initial value problem:

$$y'' + y = 1, \qquad \text{with } y(0) = 1, \quad y'(0) = 0.$$

characteristic eqn : $r^2 + 1 = 0 \Rightarrow r = \mp i$ $\qquad y_h = c_1 \cos x + c_2 \sin x$

$Y_p = A \Rightarrow 0 + A = 1 \Rightarrow Y_p = 1$

So, general solution is in the form: $y = c_1 \cos x + c_2 \sin x + 1$.

$y(0) = 1 \Rightarrow c_1 + 1 = 1 \Rightarrow c_1 = 0$

$y'(0) = 0 \Rightarrow c_2 = 0$

Thus, $y = 1$.

~~6~~ 7. (7+7+7pts) For the below differential equations write: (1) the real solution to the associated homogeneous equation y_h, and (2) the **form** of the particular solution Y_p.

(DO NOT SOLVE FOR THE COEFFICIENTS OF Y_p)

a. $y'' - 2y' + y = e^t + e^{-t}$.

characteristic eqn : $r^2 - 2r + 1 = 0 \Rightarrow r_1 = r_2 = 1$.

(1) $y_h =$ $c_1 e^t + c_2 t e^t$

(2) $Y_p =$ $t^2 A e^t + B e^{-t}$

b. $y'' - 2y' - 3y = te^{-t} - e^{-t} + 1$

characteristic eqn : $r^2 - 2r - 3 = 0$ $\quad r_1 = 3, \ r_2 = -1$

(1) $y_h =$ $c_1 e^{-t} + c_2 e^{3t}$

(2) $Y_p =$ $t(A_1 t + A_0) e^{-t} + B_0$

c. $y'' - 2y' + 5y = t^2 + 2te^{2t}\sin(t)$

characteristic eqn : $r^2 - 2r + 5 = 0$ $\quad r_1 = 1+2i, \ r_2 = 1-2i$

(1) $y_h =$ $e^t (c_1 \cos(2t) + c_2 \sin(2t))$

(2) $Y_p =$ $e^{2t}[t(A_1 \cos t + B_1 \sin t) + (A_0 \cos t + B_0 \sin t)] + D_2 t^2 + D_1 t + D_0$

Also OK: $(A_1 t + A_0) e^{2t} (C_1 \sin t + C_2 \cos t) + \cdots$

8. (13pts) Consider the differential equation

$$xy'' - (x+1)y' + y = 0$$

The function $y_1 = e^x$ is one solution. Find the general solution.

$$\begin{vmatrix} e^x & y_2 \\ e^x & y_2' \end{vmatrix} = e^{-\int -\frac{x+1}{x}\,dx} = e^{x+\ln x}$$

Abel's Th.

$$\Rightarrow \quad e^x y_2' - e^x y_2 = xe^x$$

$$y_2' - y_2 = x \quad \Rightarrow \quad \mu = e^{\int -1\,dx} = e^{-x}$$

$$y_2 = \frac{\int e^{-x} \cdot x\,dx}{e^{-x}} = \frac{-(x+1)e^{-x}}{e^{-x}}$$

$$y_2 = -(x+1)$$

$$y = c_1 e^x - c_2(x+1)$$

Alternatively, $y = ve^x$

$$y' = (v'+v)e^x$$

$$y'' = (v''+2v'+v)e^x$$

$$\Rightarrow x(v''+2v'+v)e^x - (x+1)(v'+v)e^x + ve^x = 0$$

$$\Rightarrow \quad xe^x v'' + (x-1)e^x v' = 0$$

$$\Rightarrow \quad (v')' + \left(\frac{x-1}{x}\right)v' = 0 \quad \Rightarrow \mu = e^{\int \frac{x-1}{x}\,dx} = \frac{e^x}{x}$$

$$v' = \frac{\int \frac{e^x}{x} \cdot 0\,dx}{\frac{e^x}{x}} = cxe^{-x}$$

$$\Rightarrow v = \int cxe^{-x}\,dx = -C(x+1)e^{-x}$$

$$\Rightarrow y_2 = -C(x+1)$$

$$y = c_1 e^x - c_2(x+1)$$