METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY MIDTERM 1	
Code : *MAT 119* Acad.Year: *2013-2014* Semester : *SUMMER* Date : *21.7.2014* Time : *18:40* Duration : *120 minutes*	Last Name: Name : KEY Department: Signature: Student No.: Section: 6 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1. (15)	2. (15)	3. (15)	4. (15)	5. (15)	6. (30)		

Show your work! Please draw a $\boxed{\text{box}}$ around your answers!

1. *(3× 5pts)* Compute the following limits. Do NOT use L'Hospital Rule!

(a) $\lim_{x\to 0} \tan\left(\frac{\sin \pi x}{3x}\right) \underset{\uparrow}{=} \tan\left(\lim_{x\to 0} \frac{\sin \pi x}{3x}\right) = \tan\left(\lim_{x\to 0} \frac{\pi}{3} \cdot \frac{\sin \pi x}{\pi x}\right)$

continuity of tangent

$= \tan \frac{\pi}{3} = \sqrt{3}$

(b) $\lim_{x\to -\infty} \frac{\sqrt{x^2+x+1}-x}{x+1} = \lim_{x\to -\infty} \frac{|x|\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}-x}{x+1}$

$= \lim_{x\to -\infty} \frac{\cancel{x}\left(-\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}-1\right)}{\cancel{x}\left(1+\frac{1}{x}\right)}$

$= -2$

(c) $\lim_{x\to 0} \frac{1-\cos x}{|x|} = \lim_{x\to 0} \frac{1-\cos x}{|x|} \cdot \frac{1+\cos x}{1+\cos x} = \lim_{x\to 0} \frac{\sin^2 x}{|x|.(1+\cos x)}$

$= \lim_{x\to 0} \frac{\sin^2 x}{|x|^2} \cdot \frac{|x|}{1+\cos x} = \lim_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \frac{|x|}{1+\cos x}$ $\to 1$ $\to 0$

$= 0.$

2. *(3× 5pts)* Calculate the following derivatives.

(a) $\frac{d}{dx}\left[(x^2+1)\sin(2x-1)\right] = 2x \cdot \sin(2x-1) + (x^2+1) \cdot \cos(2x-1) \cdot 2$

by product rule
and chain rule

(b) $\frac{d}{dx}\left[\frac{(x+1)\tan(2x+1)}{(3x+1)\cot(4x+1)}\right] = \frac{[(x+1)\tan(2x+1)]'\,[(3x+1)\cot(4x+1)] - [(x+1)\tan(2x+1)]\,[(3x+1)\cot(4x+1)]'}{[(3x+1)\cot(4x+1)]^2}$

by quotient rule
and chain rule

$$= \frac{[1 \cdot \tan(2x+1) + (x+1)\sec^2(2x+1) \cdot 2][(3x+1)\cot(4x+1)] - [(x+1)\tan(2x+1)][3 \cdot \cot(4x+1) - (3x+1)\csc^2(4x+1) \cdot 4]}{[(3x+1)\cot(4x+1)]^2}$$

(c) $\frac{d}{dx}\left(\sqrt{x^3+\sqrt{x^2+\sqrt{x}}}\right) = \frac{1}{2\sqrt{x^3+\sqrt{x^2+\sqrt{x}}}} \cdot \left(3x^2 + \frac{1}{2\sqrt{x^2+\sqrt{x}}} \cdot \left(2x + \frac{1}{2\sqrt{x}}\right)\right)$

3. *(15pts)* Find the equation of the tangent to the curve $x^2y - xy + xy^2 = 1 - x - y$ at (1,-1).

By taking the derivative of whole eqn;

$$2x \cdot y + x^2 \cdot y' - 1 \cdot y - xy' + 1 \cdot y^2 + x \cdot 2yy' = -1 - y'$$

$$\Rightarrow y' = -\frac{2xy - y + y^2 + 1}{2xy - x + x^2 + 1} \Rightarrow m = y'\Big|_{(1,-1)} = -\frac{2 \cdot 1 \cdot (-1) - \cancel{(-1)} + (-1)^2 + \cancel{1}}{2 \cdot 1 \cdot (-1) - \cancel{1} + 1^2 + \cancel{1}}$$

$$m = +1$$

Now, we also know that (1,-1) is on the line; so the line eqn:

$$\frac{y - (-1)}{x - 1} = +1 \Rightarrow y = x - 2$$

4. *(15 pts)* Find the alsolute maximum and absolute minimum values of $f(x) = \cos 2x + 2\cos x + 1$ on $[\frac{\pi}{2}, \frac{3\pi}{2}]$.

$$f(x) = 2\cos^2 x - \cancel{1} + 2\cos x + \cancel{1} \Rightarrow f'(x) = 4\cos x . (-\sin x) - 2\sin x$$

$$f'(x) = -2\sin x \,(2\cos x + 1)$$

Since $f'(x)$ is defined everywhere, to find critical points we need to solve: $-2\sin x\,(2\cos x + 1) = 0$

$\sin x = 0 \Rightarrow x = \pi . k$ but $x = \pi$ is only the one in $[\frac{\pi}{2}, \frac{3\pi}{2}]$

OR

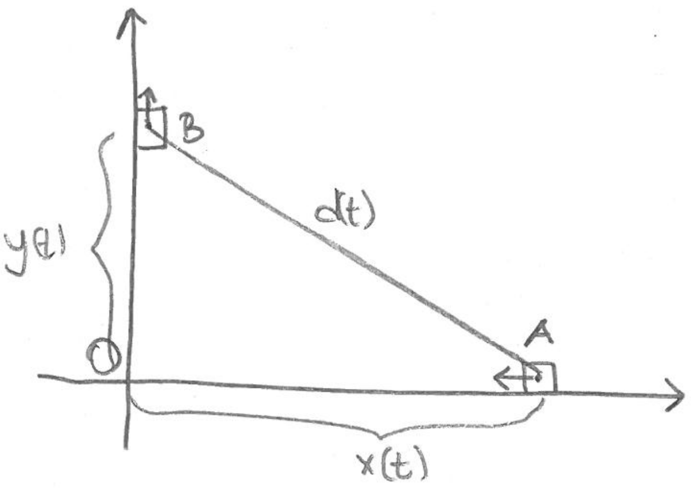
$2\cos x + 1 = 0 \Rightarrow x = \frac{2\pi}{3} + 2\pi k$; $x = \frac{4\pi}{3} + 2\pi k$.

Now to find abs max & min we list all critical points and end points:

critical points $\begin{cases} f(\pi) = 0 \\ f(\frac{2\pi}{3}) = -\frac{1}{2} \\ f(\frac{4\pi}{3}) = -\frac{1}{2} \end{cases}$ → f has abs min at $x = \frac{2\pi}{3}$ and at $x = \frac{4\pi}{3}$

end points $\begin{cases} f(\frac{\pi}{2}) = 0 \\ f(\frac{3\pi}{2}) = 0 \end{cases}$ → f has abs. max at $x = \pi$, $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$

5. *(15pts)* While car A is coming from the east, car B is going to the north. When car A is 80 km away and car B 60 km away from the origin the distance between them is de~~in~~creasing by the rate of 14 km/h. However, at that moment the area of the triangle AOB is not changing. How fast is the circumference of the triangle AOB changing at that moment?



At that moment $t = t_s$;

$x(t_s) = 80$; $y(t_s) = 60$

$d(t_s) = \sqrt{x^2(t_s) + y^2(t_s)} = 100$

$d'(t_s) = -14$ km/h

Area of AOB: $A(t) = \dfrac{x(t)y(t)}{2} \Rightarrow A'(t) = \dfrac{x'(t)y(t) + x(t)y'(t)}{2}$

since $A'(t_s) = 0 \Rightarrow x'(t_s)y(t_s) + x(t_s)y'(t_s) = 0$

$\Rightarrow 60x'(t_s) + 80\,y'(t_s) = 0 \qquad (*)$

$d^2(t) = x^2(t) + y^2(t) \Rightarrow \cancel{2}d(t_s)d'(t_s) = \cancel{2}x(t_s)x'(t_s) + \cancel{2}y(t_s)y'(t_s)$

$\Rightarrow 100 \cdot -14 = 80 \cdot x'(t_s) + 60 \cdot y'(t_s) \quad (**)$

Using $*$ in $**$; $-1400 = 80\,x'(t_s) + 60 \cdot -\dfrac{60\,x'(t_s)}{80} \Rightarrow x'(t_s) = -40$

$\Rightarrow y'(t_s) = 30$

Circumference of AOB: $C(t) = x(t) + y(t) + d(t)$

$\Rightarrow C'(t_s) = x'(t_s) + y'(t_s) + d'(t_s)$

$\Rightarrow C'(t_s) = -40 + 30 - 14 = -24$ km/h

6. *(4+4+4+6+6+6=30 pts)* Let $f(x) = \dfrac{x^3+1}{x^2-1}$. By following these steps sketch the graph of $f(x)$.

(a) Find the domain, x-intercepts and y-intercept of $f(x)$.

$D_f = \mathbb{R} - \{-1, 1\}$; $y_{int} = \dfrac{0^3+1}{0^2-1} = -1$; x_{int} : $x^3+1=0 \Rightarrow x=-1$ but not in D_f.

(b) Find ALL of the asymptotes of $f(x)$.

$$f(x) = \frac{x^3+1}{x^2-1} = \frac{(x+1)(x^2-x+1)}{(x+1)(x-1)} \underset{\text{if } x\neq -1}{=} \frac{x^2-x+1}{x-1} = x + \frac{1}{x-1}$$

Vertical Asymp: $x=1$

$\lim\limits_{x\to 1^-} f(x) = -\infty$

$\lim\limits_{x\to 1^+} f(x) = +\infty$

No horizontal Asymptote ($\lim\limits_{x\to\mp\infty} f(x) = \mp\infty$)

There is a slant asymptote

$y=x$ is the slant asymptote

(c) Check the symmetry and periodicity.

$f(-x) = \dfrac{(-x)^3+1}{(-x)^2-1} = \dfrac{-x^3+1}{x^2-1} \neq \mp f(x)$ No symmetry

Since it does not have any trigonometric func, it cannot be periodic

(d) Find the intervals of increase/decrease and local max/min points of $f(x)$.

$$f'(x) = \frac{(2x-1)(x-1) - 1(x^2-x+1)}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2}$$

f is increasing on : $(-\infty,-1) \cup (-1,0) \cup (2,\infty)$

f is decreasing on : $(0,1) \cup (1,2)$

f has local max at $x=0$

f has local min at $x=2$

x		-1		0		1		2	
$f'(x)$	+		+		−		−		+
$f''(x)$	−		−		−		+		+

(e) Find the intervals of concavity and inflection points of $f(x)$.

$$f''(x) = \frac{(2x-2)(x-1)^2 - 2(x-1)\cdot(x^2-2x)}{(x-1)^4}$$

$$f''(x) = \frac{2(x-2)\left(\cancel{x^2}-\cancel{2x}+1-\cancel{x^2}+\cancel{2x}\right)}{(x-1)^4} = \frac{2}{(x-1)^3}$$

f is concave up on : $(1,\infty)$

f is concave down on: $(-\infty,-1) \cup (-1,1)$

f has no inflection point ($x=-1$ is not in D_f!)

(f) Sketch the graph of $f(x)$.