METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY
MIDTERM 1

Code : *MAT 119*	Last Name:
Acad.Year: *2013-2014*	Name :
Semester : *FALL*	Student # :
Date : *02.11.2013*	Signature :
Time : *9:40*	6 QUESTIONS ON 5 PAGES
Duration : *110 min*	TOTAL 100 POINTS

1. (12)	2. (6)	3. (24)	4. (20)	5. (18)	6. (20)	

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(2× 6pts)* Compute the following limits. DO NOT USE L'HOSPITAL!

(A) $\lim\limits_{x\to 2} \dfrac{x^3-8}{x^2-4}$

Note that plugging in $x=0$ in $\frac{x^3-8}{x^2-4}$ results in "$\frac{0}{0}$". Therefore, Both numerator, x^3-8, and denominator, x^2-4, should have a factor of $x-2$. Incidentally, $x^3-8=(x-2)(x^2+2x+4)$ and $x^2-4=(x-2)(x+2)$.

$$= \lim_{x\to 2}\frac{x^3-8}{x^2-4} = \lim_{x\to 2}\frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x+2)} = \frac{4+4+4}{2+2} = \boxed{3}$$

Recall x is never 2 in this limit, hence $x-2$ never 0.

(B) $\lim\limits_{x\to 0} \dfrac{|\sin x|}{x}$

DOES NOT EXIST!

Here is why:

$$\frac{|\sin x|}{x} = \begin{cases} \left|\frac{\sin x}{x}\right| & \text{if } x>0 \\ -\left|\frac{\sin x}{x}\right| & \text{if } x<0 \end{cases}$$

Also recall that

(I) $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1 = \lim\limits_{x\to 0^{\pm}}\frac{\sin x}{x}$

(II) If $f(x)$ is a continuous function at $x=a$ and $\lim\limits_{x\to a} g(x) = L$, then

$$\lim_{x\to a} f(g(x)) = f(L) = f\left(\lim_{x\to a} g(x)\right)$$

(III) By (I) and (II),

$$\lim_{x\to 0}\left|\frac{\sin x}{x}\right| = \left|\lim_{x\to 0}\frac{\sin x}{x}\right| = |1| = 1.$$

(IV) $\lim\limits_{x\to 0^+}\frac{|\sin x|}{x} = \lim\limits_{x\to 0^+}\left|\frac{\sin x}{x}\right| = 1$

(V) $\lim\limits_{x\to 0^-}\frac{|\sin x|}{x} = \lim\limits_{x\to 0^-}\left(-\left|\frac{\sin x}{x}\right|\right) = -1$

(VI) Left limit and right limit are not equal, hence NO LIMIT.

2. *(6pts)* Suppose that $-x^3 \leqslant f(x) \leqslant x^3$ for all x. Show that $\lim\limits_{x\to 0}\dfrac{f(x)}{x^2} = 0$.

We apply Squeeze Theorem to get $\lim\limits_{x\to 0}\frac{f(x)}{x^2} = 0$.

(1) We must find two more functions to bound $\frac{f(x)}{x^2}$ from above and below:

For $x \neq 0$, divide

$$-x^3 \leq f(x) \leq x^3$$

by x^2 to obtain ($x^2>0$!!)

$$-x \leq \frac{f(x)}{x^2} \leq x.$$

(2) Both $\lim\limits_{x\to 0} -x$ and $\lim\limits_{x\to 0} x$ are equal to 0!

(3) Therefore Squeeze Theorem applies to $-x$, $\frac{f(x)}{x^2}$, and x: Limits of $-x$, $\frac{f(x)}{x^2}$ and x must all be the same:

$$\lim_{x\to 0}\frac{f(x)}{x^2} = 0!$$

3. *(4×6pts)* Calculate the following derivatives.

(A) $\frac{d}{dx}\left(x^3 \sin(x) \tan(x)\right)$

This is a triple product.

We apply product rule: $(f \cdot g \cdot h)' = f'gh + fg'h + fgh'$.

$$= 3x^2 \sin(x) \tan(x) + x^3 \cos(x) \tan x + x^3 \sin x \underbrace{(1+\tan^2 x)}_{= \sec^2 x}$$

(B) $\frac{d}{dx}\left(\frac{\tan(x)}{x^2+x+1} + 100\right)$

Apply quotient rule to $\frac{\tan x}{x^2+x+1}$!

$$= \frac{(1+\tan^2 x)(x^2+x+1) - \tan(x)\,(2x+1)}{(x^2+x+1)^2} + 0 \quad (\text{for } 100)$$

Derivatives of $\tan x$, x^2+x+1 and 100!

(C) $\frac{d}{dx}\left(\sqrt{x^2+\sqrt{x^2+1}}\right)$

Recall $\sqrt{x} = x^{1/2}$, $\frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-\frac{1}{2}}$. Apply Chain Rule REPEATEDLY!

$$\frac{d}{dx} u^{\frac{1}{2}} = \frac{1}{2} u^{-\frac{1}{2}} u'$$

$$= \frac{d}{dx}\left\{\left[x^2 + (x^2+1)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}$$

$$= \frac{1}{2}\left[x^2+(x^2+1)^{\frac{1}{2}}\right]^{-\frac{1}{2}}\left[2x + \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)\right], \text{ or}$$

$$= \frac{1}{2\sqrt{x^2+\sqrt{x^2+1}}} \cdot \left[2x + \frac{2x}{2\sqrt{x^2+1}}\right].$$

(D) Find y' at the point $(1,2)$ on the curve $x^2y - xy^2 = 1 - x - y$.

We want to calculate $y' = \frac{dy}{dx}$ at $(1,2)$.

(1) We treat as y as a function of x and use implicit differentiation to calculate y'!

$$x^2 y - xy^2 = 1 - x - y$$

becomes

$$2x\,y + x^2 y' - y^2 - x\,2yy' = 0 - 1 - y'.$$

after differentiation!

(2) We now solve for y':

$$(x^2 - 2xy + 1)y' + 2xy - y^2 + 1 = 0$$

gives us

$$y' = -\frac{2xy - y^2 + 1}{x^2 - 2xy + 1}$$

(3) Evaluate y' at $(1,2)$:

$$y' = -\frac{2 \cdot 2 - 4 + 1}{1 - 4 + 1} = -\frac{1}{-2} = \frac{1}{2}$$

4. *(2× 10pts)* Give the equations for the following tangent lines.

(A) The tangent line at $x = 4$ to the following function.

$$f(x) = \sqrt{9 + x^2}$$

To calculate the tangent line to the curve $y = f(x) = \sqrt{9+x^2}$ at $x=4$ we need the point and the slope.

(I) Point: Calculate $f(4) = \sqrt{9+4^2} = \sqrt{9+16} = 5$. Our point is $(4, f(4)) = (4$

(II) Slope: First calculate y' which is $\dfrac{2x}{2\sqrt{x^2+9}} = \dfrac{x}{\sqrt{x^2+9}}$.

So, $y'(4) = \dfrac{4}{\sqrt{4^2+9}} = \dfrac{4}{5}. = \text{SLOPE}.$

(III) Tangent Line: $\boxed{y - 5 = \frac{4}{5}(x-4).}$

(B) The tangent line at $(1/4,\ 2)$ to the following curve.

$$y \sin(\pi\, xy) = 8x \cos(\pi\, x)$$

Point: $\left(\frac{1}{4}, 2\right)$ Let's verify that point is on the curve.

$2\left(\sin\left(\pi \frac{1}{4} \cdot 2\right)\right) \qquad 8\frac{1}{4}\cos(\pi \cdot 2)$

$2 \cdot 1 = 2 \cdot 1.$

Slope: We need y' again. We will use implicit differentiation.

Differentiate $y \sin(\pi xy) = 8x\cos(\pi x)$:

$$y' \sin(\pi xy) + y\cos(\pi xy)\,\pi\{y + xy'\} = 8\{\cos(\pi x) + x(-\sin \pi x)\,\pi\}$$

Let's evaluate with $x = \frac{1}{4}$ and $y = 2$. We treat y' as an UNKNOWN!

Note $\pi xy = \pi \frac{2}{4} = \frac{\pi}{2}$ and $\pi x = \frac{\pi}{4}$.

$$\underbrace{y' \sin\left(\tfrac{\pi}{2}\right)}_{=1} + \underbrace{2\cos\left(\tfrac{\pi}{2}\right)}_{=0}\pi\left\{2 + \tfrac{y'}{4}\right\} = 8\left\{\cos\tfrac{\pi}{4} + \tfrac{1}{4}\left(-\sin\tfrac{\pi}{4}\right)\pi\right\}$$

$\sin\left(\frac{\pi}{2}\right) = 1$

$\cos\frac{\pi}{2} = 0$

$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = 1$

$$y' = 8\left\{\frac{\sqrt{2}}{2} - \frac{\pi}{4}\frac{\sqrt{2}}{2}\right\} = \frac{8\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right)$$

Here is y' at $\left(\frac{1}{2}, 4\right)$. $\longrightarrow = \sqrt{2}(4 - \pi) = 4\sqrt{2} - \pi\sqrt{2}$

TANGENT LINE:

$$\boxed{\begin{aligned} y - 4 &= \sqrt{2}(4-\pi)\left(x - \tfrac{1}{2}\right), \text{ or} \\ &= (4\sqrt{2} - \pi\sqrt{2})(x - 1/2) \end{aligned}}$$

5. *(18pts)* Suppose that the functions $f(x)$ and $g(x)$ are continuous on the interval $[a,b]$ and that

- $f(a) < g(a)$,
- $f(b) > g(b)$.

Show that there is $c \in [a,b]$ such that $f(c) = g(c)$. (Hint: Since $f(x)$ and $g(x)$ are continuous the function $f(x) - g(x)$ is also continuous on $[a,b]$.)

We will use Intermediate Value Theorem in this problem which states that

"If $f: [a,b] \to \mathbb{R}$ is a continuous function, and d is a real number between $f(a)$ and $f(b)$ (i.e. either $f(a) \le d \le f(b)$ or $f(a) \ge d \ge f(b)$) there is some $c \in [a,b]$ so that $f(c) = d$!

HERE IS AN ALTERNATE VERSION:

"If $h: [a,b] \to \mathbb{R}$ is a continuous function so that $h(a)$ and $h(b)$ have opposite signs, then there is some $c \in [a,b]$ so that $h(c) = 0$."

Let $h(x) = f(x) - g(x)$ (or $h(x) = g(x) - h(x)$ if you like.)

As pointed in the hint, $h(x)$ is a continuous function on the interval $[a,b]$

Since $f(a) < g(a)$ and $f(b) > g(b)$, we see that $h(a) < 0$ and $h(b) > 0$; therefore have opposite signs.

Consequently by Intermediate value theorem, there is some $c \in [a,b]$ so that

$$h(c) = 0!$$

<u>NOTE</u>: If you choose, $h(x) = g(x) - f(x)$ what changes in the above calculation is $h(a) > 0$ and $h(b) < 0$ still of opposite signs!

The rest of the argument is the same!

6. *(20pts)* A particle moves on the hyperbola $x^2 - 18y^2 = 9$ such that its y-coordinate increases at a constant rate of 9 units per second. How fast is the x-coordinate changing when $x = 9$?

This is a typical Related Rates problem. IT HAS TWO SOLUTIONS!!

GIVEN:
- A particle moving hyperbola $x^2 - 18y^2 = 9$
- y-coordinate is increasing 9 units per second, i.e.

$$\frac{dy}{dt} = +9 \text{ units/second}$$

WANTED:
- Rate of change in x-coordinate when $x = 9$ units.

i.e. $$\left.\frac{dx}{dt}\right|_{x=9} = ?$$

SOLUTION: 1) Differentiate $x^2 - 18y^2 = 9$ with respect to time:

We get $$2x\frac{dx}{dt} - 18 \cdot 2y\frac{dy}{dt} = 0$$

2) Solve the above equation for $\frac{dx}{dt}$:

$$\frac{dx}{dt} = \frac{2 \cdot 18 y \frac{dy}{dt}}{2x} = \frac{18y\frac{dy}{dt}}{x}$$

3) We want to evaluate $\frac{dx}{dt}$ for $x = 9$ units.

We also know that $\frac{dy}{dt} = 9$ units/sec.

Only unknown term above is y when $x = 9$.

4) Figure out y for $x = 9$ using $x^2 - 18y^2 = 9$

If $x = 9$; $9^2 - 18y^2 = 9 \Rightarrow 18y^2 = 81 - 9 = 72$

$y^2 = 4$

Therefore $y = 2$ or $y = -2$.

5) Answer to $\left.\frac{dx}{dt}\right|_{x=9}$ is two-fold.

a) if $y = 2$, then $\frac{dx}{dt} = \frac{18y\frac{dy}{dt}}{x} = \frac{18 \cdot 2 \cdot \cancel{9}}{\cancel{9}} = 36$ units/second.

b) if $y = -2$, then $\frac{dx}{dt} = \frac{18y\frac{dy}{dt}}{x} = \frac{18(-2)\cancel{9}}{\cancel{9}} = -36$ units/second.