METU – NCC
Mathematics Group

	Calculus with Analytic Geometry Final Exam	
Code : MATH 119 Acad.Year : 2011 Semester : Fall Coord. : S.D/I.U/H.T.	Last Name : Name : Dept. : Signature :	Stud. No : Sec. No :
Date : 13.01.2012 Time : 9.00 Duration : 180 minutes	8 Questions on 8 Pages Total **100** Points	

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8		

Q.1 (3 × 4 = 12 pts)

(a) Find $y' = \dfrac{dy}{dx}$ if $y = e^{\arcsin\left(\ln(x\,2^x)\right)}$.

$$y' = e^{\arcsin(\ln(x2^x))} \cdot \frac{1}{\sqrt{1-(\ln(x2^x))^2}} \cdot \frac{2^x + x.(\ln 2).2^x}{x.2^x}$$

(b) Find $y'' = \dfrac{d^2y}{dx^2}$ if $y = x^{\ln x}$.

$$\ln y = \ln(x^{\ln x}) = \ln x . \ln x = (\ln x)^2$$

$$\frac{y'}{y} = 2(\ln x).\frac{1}{x} \Rightarrow y' = 2\,x^{\ln x}.(\ln x).\frac{1}{x}$$

$$y'' = 2\left(2\,x^{\ln x}.(\ln x)\frac{1}{x}\right)(\ln x)\frac{1}{x} + 2x^{\ln x}\left(\frac{\frac{1}{x}.x - \ln x}{x^2}\right)$$

(c) Find $(f^{-1})'(2)$ if $f(x) = x^3 - x^2 + x + 1$.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$f^{-1}(2) = x \iff f(x) = 2$. We can easily see that $f(1) = 2$. Hence, $\left[f^{-1}(2)\right]' = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(1)} = \dfrac{1}{3-2+1}$

Q.2 ($4 \times 3 = 12$ **pts**) Evaluate the following limits:

(a) $\lim_{x\to\infty} \frac{\sqrt{\arctan x}}{x^2+1}$

$\lim_{x\to\infty} \arctan x = \pi/2$, so $\lim_{x\to\infty} \frac{\sqrt{\arctan x}}{x^2+1} = 0.$

(b) $\lim_{x\to 0} \frac{\int_1^{x^2+1} \ln t \, dt}{x^3}$ $\left(\frac{0}{0}\right)$ We can use L'Hospital Rule

$\underset{\text{by F.T.C}}{\overset{L'H}{=}} \lim_{x\to 0} \frac{\ln(x^2+1)\cdot 2\cancel{x}}{3x^{\cancel{2}}} = \lim_{x\to 0} \frac{\ln(x^2+1)\cdot 2}{3x}$ $\left(\frac{0}{0}\right)$

$\overset{L'H}{=} \lim_{x\to 0} \frac{\frac{2x}{x^2+1}\cdot 2}{3} = 0.$

(c) $\lim_{x\to 1^+} \tan\left(\frac{\pi}{4}x\right)^{\frac{1}{x-1}}$ (1^∞) $f(x) = \tan\left(\frac{\pi}{4}x\right)^{\frac{1}{x-1}}$

$\ln(f(x)) = \ln\left(\tan\frac{\pi}{4}x\right)\cdot\frac{1}{x-1}$

$\lim_{x\to 1^+} \ln f(x) = \lim_{x\to 1^+} \frac{\ln\left(\tan\left(\frac{\pi}{4}x\right)\right)}{x-1}$ $\left(\frac{0}{0}\right)$ $\overset{L'H}{=} \lim_{x\to 1^+} \frac{\frac{\sec^2\left(\frac{\pi}{4}x\right)\cdot\frac{\pi}{4}}{\tan\left(\frac{\pi}{4}x\right)}}{1}$

$= \frac{(\sqrt{2})^2}{1}\cdot\frac{\pi}{4} = \frac{\pi}{2}.$

(d) $\lim_{x\to 0^+} \frac{\frac{1}{x}+\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\cancel{\frac{1}{x}}\left(1+x\cdot\sin\left(\frac{1}{x}\right)\right)}{\cancel{\frac{1}{x}}}$

$-x \leq x\cdot\sin\left(\frac{1}{x}\right) \leq x$ for $x>0$. $\lim_{x\to 0^+} x = \lim_{x\to 0^+} -x = 0$

By squeeze theorem

$\lim_{x\to 0^+} 1 + x\cdot\sin\left(\frac{1}{x}\right) = 1 + 0 = 1.$

Q.3 ($5 \times 2 = 10$ pts) Given $y = f(x) = \dfrac{\ln x}{x}$.

(a) Write down the domain of f, and find its asymptotes.

$\mathrm{Dom}(f) = (0, \infty)$

$\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty$ (Vert. Asym at $x = 0$)

$\lim_{x \to \infty} \frac{\ln x}{x} \overset{L'H}{=} \lim_{x \to \infty} \frac{1/x}{1} = 0$ (Hor. asym. at $y = 0$)

(b) Find intervals of increase and decrease.

$f'(x) = \dfrac{\frac{1}{x} x - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$

Crit point at $\ln x = 1 \Rightarrow x = e$

	0		e	
$f'(x)$		+	0	−
		Inc.		Dec

(c) Find local maximum and minimum values of f if there is any.

$x = e$ is a local max, by 1^{st} derivative test

$f(e) = \frac{\ln e}{e} = \frac{1}{e}$

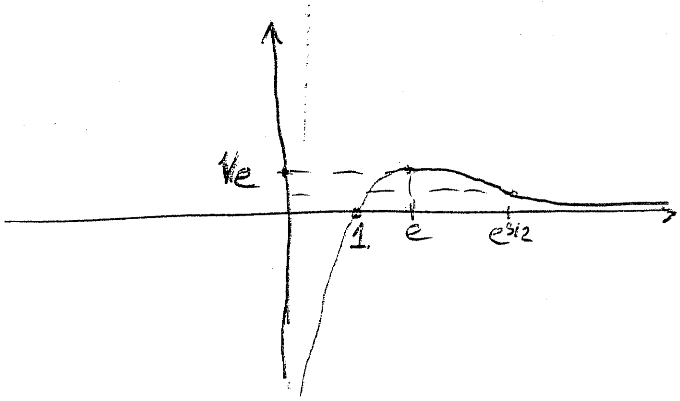
(d) Find intervals of concavity. Is there any inflection points?

$f''(x) = \dfrac{-\frac{1}{x} \cdot x^2 - (1 - \ln x) 2x}{x^4} = \dfrac{-x - (1 - \ln x) 2x}{x^4} = \dfrac{2 \ln x - 3}{x^3}$

$2 \ln x - 3 = 0 \Rightarrow \ln x = \frac{3}{2} \Rightarrow x = e^{3/2}$

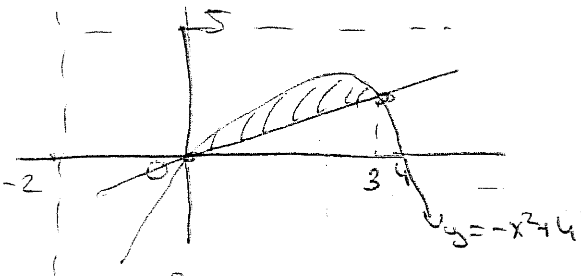
	0		$e^{3/2}$	
f''		−	0	+
		Conc Down	Inf. Point	Conc. Up

(e) Sketch its graph.



Q.4 ($4 \times 4 = 16$ pts) Let R be the region enclosed by the curves $y = -x^2 + 4x$ and $y = x$. Express the **VOLUME** of the solids (S) below as an integral. Do **NOT** evaluate the integrals.

(a) S is the solid obtained by rotating R about the x-axis.



Intersection points

$-x^2+4x = x$

$-x^2+3x = 0$

$x(-x+3) = 0$

$x=0 \qquad x=3$

$V = \int_0^3 \pi\left((-x^2+4x)^2 - x^2 \right) dx$ ← Vertical cross-section

(b) S is the solid obtained by rotating R about the y-axis.

$V = \int_0^3 2\pi x\left(-x^2+4x-x \right) dx$ ← Shell method

(c) S is the solid obtained by rotating R about the line $y = 5$.

$V = \int_0^3 \pi\left((5-x)^2 - (5-(-x^2+4x))^2 \right) dx$ ← Vertical Cross Sec

(d) S is the solid obtained by rotating R about the line $x = -2$.

$V = \int_0^3 2\pi(2+x)(-x^2+4x-x)\, dx$ ← Shell method

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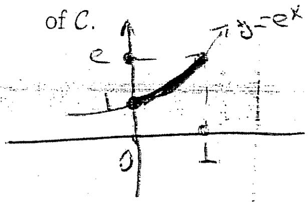
Q.5 (4 × 3 = 12 pts) Consider the plane curve C defined by $y = e^x$ between $y = 1$ and $y = e$.

(a) Set up, but do **NOT** evaluate, an integral with respect to y for the **arclength** of C.



$y = e^x \Leftrightarrow x = \ln y$

$$\int_1^e \sqrt{1 + \frac{1}{y^2}} \cdot dy$$

(b) Set up, but do **NOT** evaluate, an integral with respect to x for the **arclength** of C.

$$\int_0^1 \sqrt{1 + e^{2x}} \cdot dx$$

(c) Set up, but do **NOT** evaluate, an integral for the **area** of the **surface** obtained by rotating C about the y-axis.

$$A = \int_1^e 2\pi \ln y \sqrt{1 + \frac{1}{y^2}} \cdot dy$$

(d) Set up, but do **NOT** evaluate, an integral for the **area** of the **surface** obtained by rotating C about the x-axis.

$$A = \int_0^1 2\pi e^x \sqrt{1 + e^{2x}}\, dx$$

Q.6 ($6 \times 4 = 24$ pts) Evaluate the following integrals:

(a) $\int e^{3x} \sin(e^{3x})\, dx = \frac{1}{3}\int \sin(u)\, du = -\frac{1}{3}\cos(e^{3x}) + C$

$u = e^{3x}$

$du = 3e^{3x} \cdot dx$

(b) $\int_0^{\pi/2} \sin^3 x \cos^{12} x\, dx = \int_0^{\pi/2} \sin x (1-\cos^2 x) \cdot \cos^{12} x \cdot dx$

$u = \cos x$, $du = -\sin x \cdot dx$

$= -\int_1^0 (1-u^2) \cdot u^{12}\, du = \int_0^1 u^{12} - u^{14}\, du = \left. \frac{u^{13}}{13} - \frac{u^{15}}{15} \right|_0^1$

$= \frac{1}{13} - \frac{1}{15}$.

(c) $\int \frac{x^2-2}{x^2-2x}\, dx = \int \left(1 + \underbrace{\frac{2x-2}{x^2-2x}}\right) dx = x + \ln|x^2-2x| + C$

$u = x^2 - 2x$

$du = 2x - 2$

(d) $\int \frac{3x^2 - x + 3}{(x-1)(x^2+4)}\, dx = \int \left[\frac{1}{x-1} + \left(\frac{2x+1}{x^2+4}\right)\right] dx$

$\frac{3x^2-x+3}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} = \frac{x^2(A+B) + x(C-B) + 4A - C}{(x-1)(x^2+4)}$

$(x^2+4) \qquad (x-1)$

$A + B = 3 \longrightarrow 5A = 5 \Rightarrow A = 1$

$C - B = -1 \Rightarrow C = B - 1 \qquad \Rightarrow B = 2$

$4A - C = 3 \Rightarrow 4A - B + 1 = 3 \Rightarrow 4A - B = 2 \qquad \Rightarrow C = 1$

$\int \frac{1}{x-1}\, dx + \int \frac{2x}{x^2+4}\, dx + \int \frac{1}{x^2+4}\, dx$

$u = x^2 + 4$

$du = 2x$

$= \ln|x-1| + \ln|x^2+4| + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$

Integration by parts

(e) $\int_0^4 e^{\sqrt{x}}\, dx = 2\int_0^2 u \cdot e^{u} \cdot du \overset{\downarrow}{=} 2(ue^{u} - e^{u})\Big|_0^2 = 2(2e^2 - e^2) - 2($

$= 2e^2 + 2$

$u = \sqrt{x}$

$u^2 = x$

$2u \cdot du = dx$

(f) $\int \frac{x^2}{\sqrt{1-x^2}}\, dx = \int \frac{\sin^2\theta \cdot \cos\theta \cdot d\theta}{\underbrace{\sqrt{1-\sin^2\theta}}_{\cos\theta}} = \int \sin^2\theta \, d\theta$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$

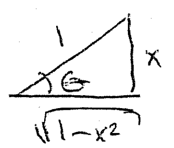
$\theta = \arcsin x$

$= \frac{1}{2}\int (1 - \cos 2\theta)\, d\theta$

$= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$

$= \frac{1}{2}\arcsin x - \frac{1}{2}\sin\theta \cdot \cos\theta$

$= \frac{1}{2}\arcsin x - \frac{1}{2} x \sqrt{1-x^2} + C$



Q.7 (5 pts) Let $A(h)$ be the area under $f(x) = e^{-x^2}$ between $x = 0$ and $x = h$. Suppose that the parameter h changes over time t with $h(t) = t^2 + t$. Find the **rate of change** of $A(h)$ at time $t = 1$.

$A(h) = \int_0^h e^{-x^2} \cdot dx$

$\frac{dA}{dt} = \frac{dA}{dh} \cdot \frac{dh}{dt} = e^{-h^2} \cdot (2t + 1)$

$\frac{dA}{dt}\Big|_{t=1} = e^{-4}(3)$

Q.8 (3 × 3 = 9 pts)

(a) Evaluate the integral $\int_0^\infty xe^{-x^2}\,dx$, if it is convergent.

$$\int x.e^{-x^2}.dx = -\frac{1}{2}\int e^{u}.du = -\frac{1}{2}e^{u} = -\frac{1}{2}e^{-x^2}+C$$

$u = -x^2$

$du = -2x.dx$

$$\int_0^\infty xe^{-x^2}.dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}.dx = \lim_{t\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^t$$

$$= \lim_{t\to\infty} -\frac{1}{2}e^{-t^2}+\frac{1}{2} = \frac{1}{2} \qquad \text{Convergent !}$$

(b) Evaluate the integral $\int_0^2 \frac{2x}{x^2-1}\,dx$, if it is convergent.

$$\int_0^2 \frac{2x}{x^2-1}.dx = \lim_{t\to 1^-}\int_0^t \frac{2x}{x^2-1}.dx + \lim_{t\to 1^+}\int_t^2 \frac{2x}{x^2-1}.dx$$

$$= \lim_{t\to 1^-} \ln|x^2-1|\Big|_0^t + \lim_{t\to 1^+} \ln|x^2-1|\Big|_t^2$$

$$= \underbrace{\lim_{t\to 1^-} \ln|t^2-1|}_{-\infty} + \lim_{t\to 1^+} \underbrace{\ln 3 - \ln|t^2-1|}_{\infty}$$

So $\int_0^2 \frac{2x}{x^2-1}\,dx$ is divergent

(c) Use a comparison test to determine whether the integral

$$\int_1^\infty \frac{\sin^2 x}{\sqrt{x}+x^2}\,dx$$

is convergent or divergent.

$0 \le \sin^2 x \le 1$ & $0 \le \sqrt{x}$, so $\frac{\sin^2 x}{\sqrt{x}+x^2} \le \frac{1}{x^2}$

$$\int_1^\infty \frac{1}{x^2}.dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}.dx = \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t = \lim_{t\to\infty}\left(-\frac{1}{t}+1\right) =$$

By comparison test $\int_1^\infty \frac{\sin^2 x}{\sqrt{x}+x^2}$ is also convergent

since

$$\int_1^\infty \frac{\sin^2 x}{\sqrt{x}+x^2}\,dx \le \underbrace{\int_1^\infty \frac{1}{x^2}\,dx}_{\text{convergent}}$$