METU - NCC

LINEAR ALGEBRA MIDTERM EXAM 2	
Code : *MAT 260* Acad.Year: *2013-2014* Semester : *SPRING* Date : *10.05.2014* Time : *13:40* Duration : *90 min*	Last Name: Solution KEY Name : Student # : Signature : 5 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1. (20)	2. (20)	3. (20)	4. (20)	5. (20)			

1. *(20pts)* Let $T : \mathbb{R}^3 \to \mathcal{P}_2$ be the linear transformation

$$T(a, b, c) = (a + b) + (a + c)x + (b + c)x^2.$$

Show that T is an isomorphism.

Note that $\vec{u} = (a, b, c) \in \ker(T)$ iff

$$\begin{cases} a+b=0 \\ a+c=0 \\ b+c=0 \end{cases} \iff a=b=c=0 \text{ or } \vec{u}=\vec{0}.$$

Hence $\ker(T) = \{0\}$. But $\dim(\mathbb{R}^3) = \dim(P_2) = 3$.

By Dimension Formula, $\dim(\operatorname{im}(T)) = 3$, that is,

T is an isomorphism.

2. *(10+20pts)* Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be the linear transformation $T(x, y) = (x + y, x - y, 2x, 2y)$.

(a) Find the matrix representation of T with respect to the pair of basis $\{(1, 0), (0, 1)\}$ $\overset{\shortparallel}{e}$ and $\{(-1, -1, 0, 0), (1, -1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$. $= f$

$$M_{(e,f)}(T) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}, \text{ for } \begin{array}{l} Te_1 = -f_1 + 2f_3 \\ Te_2 = f_2 + 2f_4 \end{array}$$

(b) Find the matrix representation of T with respect to the pair of basis $\{(1, 1), (1, -1)\}$ and $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ USING THE BASE CHANGE METHOD from the matrix you found in part (a).

Put $e' = \{(1,1),(1,-1)\}$ and $f' = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$. Using Base Change Method, we derive that

$$M_{(e',f')}(T) = \underset{\shortparallel}{M_{(f,f')}(\mathbb{1})}\; \underset{\shortparallel}{M_{(e,f)}(T)}\; \underset{\shortparallel}{M_{(e',e)}(\mathbb{1})} =$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 2 & 2 \\ 2 & -2 \end{bmatrix}$$

3. *(20pts)* Let V be the plane defined by the equation $x + 2y - z = 0$ in the space $\mathbb{R}^3$. Find the matrix $M_{(e,e)}(P)$ of the skew projection $P : \mathbb{R}^3 \to \mathbb{R}^3$ onto the plane V parallel to the vector $(0,0,1)$ with respect to the standard basis e.

Take a basis for V, say $f_1 = (2,-1,0)$, $f_2 = (1,0,1)$. Then $f = (f_1, f_2, f_3)$ with $f_3 = (0,0,1)$ is a basis for $\mathbb{R}^3$ and $M_{(f,f)}(P) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. But

$$M_{(e,e)}(P) = M_{(f,e)}(\mathbb{1})\, M_{(f,f)}(P)\, M_{(e,f)}(\mathbb{1})$$ and

$$M_{(f,e)}(\mathbb{1}) = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$ To find $M_{(e,f)}(\mathbb{1})$ we have to express e in terms of f. Namely,

$e_1 = f_2 - f_3$, $e_2 = -f_1 + 2f_2 - 2f_3$, $e_3 = f_3$, that is,

$$M_{(e,e)}(P) = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 2 & 0 \\ -1 & -2 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 2 & 0 \\ -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

Test: $M_{(e,e)}(P)^2 = M_{(e,e)}(P)$!

4. *(10pts)* Suppose that $T : V \to V$ is a linear transformations such that $T^2 = I$. Show that $\frac{1}{2}(T + I)$ is a projection.

Put $P = \frac{1}{2}(T + I)$. Then $P^2 = \frac{1}{4}(T+I)(T+I)$

$= \frac{1}{4}(T^2 + 2T + I) = \frac{1}{4}(2T + 2I) = \frac{1}{2}(T + I)$

$= P$, that is, $P^2 = P$.

5. *(20pts)* Determine whether or not the matrices below represent the same linear transformation with respect to different pairs of basis. Explain your answer.

$$M_{(e,f)}(T) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = M_{(e',f')}(S)$$

Note that (Te_1, Te_2, Te_3) is a linearly independent vector, thereby $\dim(\operatorname{im}(T)) = 3$. But

$$Se_3' - Se_1' = 3f_1' + f_2' + f_3' - f_1' - f_2' - f_3' = 2f_1' = Se_2',$$

that is, $\dim(\operatorname{im}(S)) = 2$.

Whence $T \neq S$.