METU - NCC

LINEAR ALGEBRA MIDTERM EXAM 1	
Code : *MAT 260* Acad.Year: *2013-2014* Semester : *SPRING* Date : *12.04.2014* Time : *13:40* Duration : *90 min*	Last Name: Name : Student # : Signature : 5 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1. (20)	2. (20)	3. (20)	4. (20)	5. (20)			

1. *(10+10pts)* Let $E = \{(1,0,-1),(2,1,0)\}$ be vectors in $\mathbb{R}^3$.

(a) Show that E is linearly independent.

$$\lambda(1,0,-1) + \mu(2,1,0) = \vec{0} \iff \begin{array}{l} \lambda + 2\mu = 0 \\ \mu = 0 \\ -\lambda = 0 \end{array}$$

That is $\lambda = \mu = 0$.

(b) Find a basis of $\mathbb{R}^3$ containing E.

Take $(0,1,0)$ as a third vector. Then

$$\lambda(1,0,-1) + \mu(2,1,0) + \theta(0,1,0) = \vec{0} \iff$$

$$\begin{cases} \lambda + 2\mu = 0 \\ \mu + \theta = 0 \\ -\lambda = 0 \end{cases} \Rightarrow \lambda = \mu = \theta = 0.$$

2. *(20pts)* Let $S = \{a, b, c, d\}$ and consider the following subspace of $Fun(S)$:

$$U = \{f \in Fun(S) : \quad f(a) + f(b) + f(c) = 0, \text{ and } f(c) + 2f(d) = 0\}$$

Find a basis of U.

We can use the isomorphism $Fun(S) \to \mathbb{R}^4$, $x_i \mapsto e_i, 1 \le i \le 4$.

Then $\mathcal{U} = \{ x+y+z=0, \ z+2w=0 \}$ up to an isomorphism.

Take $\vec{u} = (x, y, z, w) \in \mathcal{U}$. Then $z = -x-y$, $w = -\frac{1}{2} z = +\frac{1}{2}(x+y)$

$\vec{u} = (x, y, -x-y, +\frac{1}{2}(x+y)) = x\,(1, 0, -1, +\frac{1}{2}) + y\,(0, 1, -1, +\frac{1}{2})$

$= x\, \vec{f}_1 + y\, \vec{f}_2$, where

$\vec{f}_1 = (1, 0, -1, +\frac{1}{2})$, $\vec{f}_2 = (0, 1, -1, \frac{1}{2}) \in \mathcal{U}$ are linearly independent vectors. Thus $\mathcal{U} = \mathrm{Span}\{\vec{f}_1, \vec{f}_2\}$ and $\dim(\mathcal{U}) = 2$. Turning back we obtain that

$\vec{f}_1 \equiv x_a - x_c + \frac{1}{2} x_d$, $\vec{f}_2 \equiv x_b - x_c + \frac{1}{2} x_d$,

which is a basis for $\mathcal{U}$.

3. *(10+10pts)* Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation such that

$$T(1,0,0) = (1,-1,-1)$$
$$T(0,1,0) = (0,2,1)$$
$$T(0,0,1) = (2,-2,-2)$$

(a) Determine $T(2,1,-1)$.

First note that

$$T(x,y,z) = (x,-x,-x) + (0,2y,y) + (2z,-2z,-2z) =$$

$$= (x+2z, -x+2y-2z, -x+y-2z).$$

So, $T(2,1,-1) = (0, 2, 1)$

(b) Find $Ker(T)$ (The kernel of T).

We have $\vec{a} = (x,y,z) \in \ker(T)$ iff

$$\begin{cases} x+2z=0 \\ -x+2y-2z=0 \\ -x+y-2z=0 \end{cases} \iff \begin{matrix} x+2z=0 \\ y=0 \end{matrix}$$

Whence $\ker(T) = \{x+2z=0, y=0\}$, $\dim(\ker(T)) = 1$,

$\ker(T) = \text{Span}\{(2,0,-1)\}$.

4. *(20pts)* Let U be subspace of a finite dimensional vector space V and assume that $dim(U) = dim(V)$. Prove that $U = V$.

Take a basis $e_1, \dots, e_n$ for $\mathcal{U}$. By Basis Extension Theorem, $e_1, \dots, e_n, e_{n+1}, \dots, e_m$ is a basis for V for some vectors $e_{n+1}, \dots, e_m$. By assumption $n = m = \dim(V)$. Therefore $e_1, \dots, e_n$ is a basis for V as well.

In particular,

$$V = \operatorname{Span}\{e_1, \dots, e_n\} = \mathcal{U}.$$

5. *(20pts)* Let $D : \mathcal{P}_4(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R}), D(p(x)) = p'(x)$ be the differentiation transformation. Show that $\text{im}(D) = \mathcal{P}_3(\mathbb{R})$ and find $\dim(\ker(D)) =?$. Explain your answer.

If $p(x) \in P_4(\mathbb{R})$ then $p'(x) \in P_3(\mathbb{R})$ without any doubt. Conversely, if $q(x) \in P_3(\mathbb{R})$ then

$q(x) = \frac{d}{dx} \int_0^x q(t)\, dt \in \operatorname{im}(D)$ by virtue of F.T.C.

So, $\operatorname{im}(D) = P_3(\mathbb{R})$. Finally, based on Dimension Formula, we derive that

$$5 = \dim(P_4(\mathbb{R})) = \dim(\ker(D)) + \dim(\operatorname{im}(D)) =$$

$$= \dim(\ker(D)) + 4 \Rightarrow \dim(\ker(D)) = 1.$$